# Homotopical mathematics

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# Contents

# Introduction

Throughout the history of mathematics there has been a rich interplay between synthetic and analytic methods. With synthetic, which could also be called axiomatic, we mean that the objects of interests and the relations between them are postulated and manipulated directly. This method goes back to Euclid's elements which served as a gold standard for mathematical reasoning until deep into the middle ages. The analytic methods in contrast try to understand objects through their instantiations, or models, in existing structures.

In the case of geometry it was Decartes that instantiated Euclid's geometric shapes in the real (Cartesian) plane. This model provided a link between geometric and algebraic methods which allowed later mathematicians, such as Gauß to provide solutions to geometric problems which puzzled geometers for centuries. Analytical thinking perhaps found it's apex in the beginning of the 20th century mathematics when all mathematical theories were understood through their incarnations in Cantor's set theory. This program was hugely successful by connecting all of mathematics and providing rich opportunities of cross pollination between mathematical fields. In the words of Hilbert:

> Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können. (From the paradise, that Cantor created for us, no-one shall be able to expel us.)

But the analytic method comes with it's own set of challenges. Particularly the models of objects in some ambient theory often carry spurious structure not essential to the objects studied. For example familiar mathematical objects such as topological spaces, manifolds, and vector spaces where orginally based on certain properties of cartesian spaces $\mathbb{R}^n$. Now when we represent a vector space by some cartesian $\mathbb{R}^n$ it comes with an preferred basis which allows us to make statements which are not essentially about vector spaces but particular for this representation. The situation with summed up by Norton [Nor93]:

> Our modern difficulty in reading Einstein literally actually stems from a change [..] in the mathematical tools used [..]. In recent work [..] we begin with a very refined mathematical entity, an abstract differentiable manifold [..]. We then judiciously add further geometric objects only as the physical content of the theory warrants [..]. In the 1910s, mathematical practices in physics were different. [One] used number manifolds $\mathbb{R}^n$ or $\mathbb{C}^n$ for example. Thus Minkowski's 'world' [..] was literally $\mathbb{R}^4$, that is it was the set of all quadruples of real numbers. Now anyone seeking to build a spacetime theory with these mathematical

> tools of the 1910s faces very different problems from the ones we see now. Modern differentiable manifolds have too little structure and we must add to them. Number manifolds have far too much structure [..] the origin $(0, 0, 0, 0)$ is quite different from any other point, for example [..]. The problem was not how to add structure to the manifolds, but how to deny physical significance to existing parts of the number manifolds. How do we rule out the idea that $(0, 0, 0, 0)$ represents the preferred center of the universe?

An essential tool for dealing with spurious structure is to associate to a model of a mathematical object an group of symmetries which capture the intended invariant symmetries. This means that any statement we make about the model of an mathematical object must be invariant under the group action. Felix Klein formulated this idea in his Erlangen program:

> Given a manifold and a transformation group acting on it, to investigate those properties of figures on that manifold which are invariant under [all] transformations of that group.

Cantors paradise provided the backdrop for the next spur in synthetic methods under the name of universal algebra. Various properties of the cartesian theater where abstracted away to produce mathematical objects such as topological spaces, manifolds, vector spaces and so on. All these objects instantiated in the lingua franca of sets. This painted a new kind of picture of mathematics, with all these kinds of objects having equal ontological status in the pantheon of set theory.

In fact the set theoretical framework serves *only* to legitimize these mathematical objects, since exploiting their particular encodings would be tantamount to using spurious structure. This observation provides the foundation for structuralist thinking, which states that the only legitemate way to study mathematical objects is in a context. For example the same set $\{\varnothing, \{\varnothing\}\}$ might encode both an Von Neumann natural number 2 and a topology on $\{\varnothing\}$, so without a supplied context we should assign it no meaning.

Category theory, the subject of part I of this thesis, provides a very powerful and elegant way to think structurally. Especially striking is that category theory does not set theoretic origins but instead reaffirms the central role of sets. Category theory unifies, clarifies and organizes a great deal of modern mathematics. There are still some difficulties with this paradigm, such as the irregular behaviour of some colimits as highlighted in the chapter on topoi.

We now wish to make the following, somewhat speculative, suggestion. The identification of the Cartesian plane in the 17th century was pivotal for the development of much of mathematics in the centuries to come. To be precise cartesian lines, planes and volumes served as the backdrop in which mathematical objects could be analytically understood and studied. The 19th-20th century identification of sets transported us from a cartesian theater to cantors paradise in which the lines and planes naturally find their place in a grander framework. In the process more mathematical objects could be analytically studied and compared in an holistic framework. We now suggest that we are on the cusp of the next ontological jump in mathematics replacing sets with $\infty$-groupoids or homotopy types of spaces. What exactly these objects are we will see at the end of this thesis but for now it will be enough to say that they enable the study of objects with higher symmetries.

Whether the reader wishes to indulge in the above speculation or not, the fact remains that the development of the theory of $\infty$-groupoids presents considerable difficulties in set theoretic frameworks. In fact to do this we start with topological spaces, which we speculatively assume to be models for these new kinds of objects, and then carefully work to forget the spurious information carried by this topological encoding. This project goes by the name of homotopical mathematics and is the focus of part II of this thesis.

The study of objects based in $\infty$-groupoids goes by the name of $\infty$-category theory. In a set theoretic framework these object can again only be understood through models. With the development of models and aspects of the theory by Joyal and Lurie these objects have been understood for just a few decades. Homotopy type theory, originally meant to be the final part of this thesis, is hypothesized to be an synthetic description of these objects. With the recent work of Shulman [Shu19] an important piece of this picture is filled in. One can thus hope that the tandem arrival of homotopical mathematics as model and homotopy type theory as language for these objects might spur an ontological shift similar to those produced by the cartesian theater and cantors paradise.

# Part I

# Category theory

The goal of part I is two fold, one is to lay the foundations for discussing homotopical mathematics later in part II, and the second is to build up to the theory of topoi in the last chapter of part I.

For this we first introduce the notion of category in chapter 1. This chapter begins with the introduction highlighting some of the conceptual essence of category theory. This mostly consists of rephrasing familier concepts in this new framework. Then we move on to representables and the Yoneda lemma which showcase the novel perspectives categorical thinking has to offer. Then in the next sections we highlight this by making the necessary constructions needed for in the rest of this thesis using this language.

The theme of chapter 2 is the identification and manipulation of subcategories. In the first section we meet the exceedingly important notion of a reflective subcategory. Reflective subcategories represent an ideal kind of subcategory with a strong relation to the category they are contained in. We will see that restriction to a reflective subcategory corresponds to adding inverses to some morphisms. This process of inverting morphisms is called localization and it allows us to treat some morphisms as isomorphisms. Unfortunately localization at a collection of morphisms does not always yield a reflective subcategory, we nevertheless wish to find tools to approximate this operation. Finding a context to do that is essentially the topic of part II. A necessary tool for working with localizations are the decomposition and factorization systems introduced in the last section of chapter 2.

The goal of the last chapter is to give an introduction to topos theory. To a first approximation a topos can be considered a category which behaves like the category of sets, meaning that many constructions in set translate to any topos. Categories that stand very close to sets are the presheaf categories which we will define first. Before continuing with topos theory we then define the technique of extension by colimit in 3.1, the last technical result we need for part II. Before characterizing topoi we spent section 3.2 introducing presentable categories. These provide a categorical way to deal with size issues and have some very pleasing properties. Finally we then introduce the classical notion of a topos in section 3.3. Topoi are space like objects, so by the fundamental duality between space and algebra we expect to characterize them by an algebraic object. Indeed logoi as introduced by Joyal and Anel [AJ19] capture the algebraic properties of topoi. It is then precisely the restriction to presentable logoi that are dual to topoi.

# Chapter 1

# Categories

## 1.1 Introduction to categories

**1.1.** A **category** $\mathsf{C}$ consists of a collection $\mathcal{O}\mathrm{bj}(\mathsf{C})$ of **objects** $X, Y, Z$ and a collection $\mathcal{M}\mathrm{or}(\mathsf{C})$ of **morphisms** $f, g, h$, such that

**category**

- Each morphism has specified **domain** and **codomain** objects; the notation $f : X \to Y$ indicates that $X$ is the domain of $f$ and $Y$ is the codomain, i.e. $f$ is an morphism from $X$ to $Y$.
- Two morphisms $f$ and $g$ of $\mathsf{C}$ are **composable** if the domain of $g$ is the codomain of $f$. In this case there is an morphism $g \circ f$ or just $gf$ called the **composite** of $g$ and $f$. This **composition** operation $\circ$ is moreover associative. Concretely, given $f : X \to Y$, $g : Y \to Z, h : Z \to Z'$ we have $gf : X \to Z$ and $(hg)f = h(gf)$.
- Each object has a designated **identity** morphism written $\mathsf{id}_X : X \to X$ or sometimes just $\mathsf{id}$. For any morphism $f : X \to Y$ the appropriate (i.e. composable) $\mathsf{id}_Y$ and $\mathsf{id}_X$ serve as left and right unit, i.e. $\mathsf{id}_Y f = f$ and $f\mathsf{id}_X = f$.

**1.2.** When we use the word 'collection' above we have a set in mind. The reason for using the vague term 'collection' is for essentially two different reasons.

1. The first of those is related to size: if we take a set theoretical framework like ZFC then for many categories there will be a proper class of objects/morphisms. When the classes of morphisms and objects of a category $\mathsf{C}$ are sets the category $\mathsf{C}$ is said to be **small**. Note that for every object in category there is at least one morphism, namely the identity morphism, so an category is already small if its class of morphisms is small.

   **small category**

2. The second is a subtle but central point; any set comes pre-equipped with the equivalence relation of set theoretical equality on its members. We wish to deemphasize this equality. The reasons for this are: 1) remembering equality would lead to a natural

notion of equivalence of categories based on a bijection between objects and morphisms, but this notion is too strong in practice, see 1.28; and 2) the morphisms of a category already induce an equivalence relation on its set of objects which will be the right notion of equality between objects, see 1.5.

**1.3.** For a pair of objects $X$ and $Y$ of a category $\mathsf{C}$ the **hom class**, written $\mathsf{Hom}_{\mathsf{C}}(X, Y)$ or just $\mathsf{Hom}(X, Y)$, is the class of morphisms from $X$ to $Y$, i.e. $\mathsf{Hom}_{\mathsf{C}}(X, Y) := \{f \in \mathcal{M}\mathrm{or}(\mathsf{C}) \mid f : X \to Y\}$. Many categories that fail to be small can still be **locally small** in the sense that for each $X, Y \in \mathsf{C}$ the class $\mathsf{Hom}(X, Y)$ is a set which we then call **hom set**.

**hom class**
**locally small**
**hom set**

**1.4.** Categories deserve their name for **organizing** mathematical objects of study. When looking at introductory texts one often encounters that immediately after the introduction of the objects of study (e.g. groups, topological spaces, rings) the relevant notion of structure preserving map between them is given (e.g. group homomorphism, continuous map, ring homomorphism). This is very much in the spirit of category theory and by the above definition we can collect such objects into a relevant category of which we will now give some examples. The following are examples of categories:

**organizing categories**

(i) The category $\mathcal{S}\mathrm{et}$ consisting of sets and functions between them.

(ii) The category $\mathcal{G}\mathrm{rp}$ consists of groups and group homomorphisms.

(iii) The category $\mathcal{T}\mathrm{op}$ consists of topological spaces and continuous maps.

In general, whenever we have some (essentially) algebraic theory we can consider the category of its models.

**1.5.** Arguably the reason for adding identities to the definition of a category is in order to state the following definition. A morphism in a category $f : X \to Y$ is said to be an **isomorphism** if there is an morphism $g : Y \to X$ such that $fg = \mathsf{id}_Y$ and $gf = \mathsf{id}_X$. In this case the morphism $g$ is unique and will be written $f^{-1}$. Two objects $X$ and $Y$ are said to be **isomorphic** written $X \cong Y$ if there is an isomorphism between them.

**isomorphism**
**isomorphic**

The relation of isomorphism between objects is an equivalence relation. Indeed we have $X \cong X$ by $\mathsf{id}_X$ and if $f : X \to Y$ and $g : Y \to X$ are equivalences, then so is $gf$ with inverse $(gf)^{-1} = f^{-1}g^{-1}$. The notion of equivalence in a category $\mathsf{C}$ will serve as the notion of 'equivalence' on the collection $\mathcal{O}\mathrm{bj}(\mathsf{C})$ of its objects. This will be made precise in section 5.2.

**1.6.** We would like to extend the above definition of isomorphism to the following principle. In general we will abstain from using the notion of equality of mathematical objects and instead will prefer to define them in the context of a category. Then isomorphism in the category becomes the right notion of equivalence between objects. The reason for adhering to this principle is that mathematicians have often found it useful to treat mathematical objects such as the ones mentioned in 1.4 as the identical if they are isomorphic. If we refrain from referring to equality of objects then our statement are automatically invariant under the implicit quotient we take when we treat isomorphic objects as identical. Of course in current foundations the set theoretical equality relation will always be present making it easy to slip up and or making definitions awkward to phrase. The language of homotopy type theory aims to be a language for which all statements are always invariant under equivalence.

The above principle has important repercussions if the morphisms of a certain category are part of category themselves. In this case we should not be allowed to talk about equality of morphism, instead we should talk about morphisms being isomorphic. The category of categories $\mathcal{C}\mathrm{at}$, which we will see soon, is such a category.

**1.7.** Apart from the organizing categories from 1.4 we met before it is also possible to define **abstract categories** which are defined by directly specifying a set of objects and a set of morphism. Abstract categories show that many familiar mathematical objects can be encoded in the formalism of category theory:

**abstract categories**

(i) Any monoid $M$ is category with a single object $\{*\}$, such that $\mathsf{Hom}(*,*) = M$ such that the composition is the monoid action. Conversely for any category $\mathsf{C}$ with object $X \in \mathsf{C}$ the set of $\mathsf{Hom}(X, X)$ is a monoid.

(ii) A groupoid is a category in which all morphisms are isomorphisms. Conversely any category $\mathsf{C}$ has a **core** groupoid $\mathsf{Core}(\mathsf{C})$ which has the same objects and all isomorphisms of $\mathsf{C}$.

**core**

(iii) A group is a monoid which is also a groupoid. Conversely we can consider for a category $\mathsf{C}$ with object $X \in \mathsf{C}$ the the **automorphism group** $\mathsf{Aut}(X)$ of isomorphisms in $\mathsf{Hom}(X, X)$, this is a group under composition with $\mathsf{id}_X$ as unit.

**automorphism group**

(iv) Any poset is a strict category such that every hom set is either the empty or the set with one element.

(v) Every ordinal is a category. An ordinal is a well ordered set and so in particular a poset and hence a category by (iv). For each $n \geq 0$ write $[n]$ for the linear order with $n + 1$ elements. We write the elements of $[n] := \{0, 1, ..., n\}$ where the order relation is the obvious one. Also write $[-1] = \varnothing$ for the empty linear order.

(vi) For each cardinal $\kappa$ there is a category $\mathsf{Disc}(\kappa)$ with $\kappa$ objects and only identity morphisms.

Another supply of abstract categories can be obtained by sketching a directed graph and completing it into a category in a canonical way. For example we may draw a graph missing only the identity arrows. Or we might draw a graph and freely add all identities and compositions to obtain a category.

**1.8.** Note that the definition of a category is itself also an (essentially) algebraic definition[1], so there is a notion of a structure preserving map between categories. Such maps are called functors, see below. With this observation we an expand the list of organizing categories 1.4 with.

(iv) The category $\mathcal{C}\mathrm{at}$ consisting of categories and functors.

**1.9.** A **functor** $F$ between a categories $\mathsf{C}$ and $\mathsf{D}$ consists of

**functor**

- a function $F_0 : \mathcal{O}\mathrm{bj}(\mathsf{C}) \to \mathcal{O}\mathrm{bj}(\mathsf{D})$ also written $F$; and
- a function $F_1 : \mathcal{M}\mathrm{or}(\mathsf{C}) \to \mathcal{M}\mathrm{or}(\mathsf{D})$, also written $F$

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[1] In the technical sense of[Adá+94, 3.D]

such that for a morphism $f : A \to B$ we have $F(f) : F(A) \to F(B)$ preserving composition and identities:

(i) for any composable pair $f$ and $g$ we have $F(gf) = F(g)F(f)$

(ii) for any $X \in \mathsf{C}$ we have $F(\mathsf{id}_X) = \mathsf{id}_{F(X)}$

Any functor $F$ sends isomorphisms to isomorphisms. This means in particular that the object part of $F$ respects the isomorphism relation on the collection of objects.

**1.10.** With the definition of a functor we might be tempted to ask when two categories $\mathsf{C}$ and $\mathsf{D}$ are equivalent. An equivalence would be a pair $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ such that $GF = \mathsf{id}_\mathsf{C}$ and $FG = \mathsf{id}_\mathsf{D}$, but this requires us to talk about equivalence of functors. The principle of equivalence 1.6 implies that we can do this once we know what the category of functors is.

For each pair $\mathsf{C}$ and $\mathsf{D}$ of categories there is a category $\mathcal{F}\mathrm{un}(\mathsf{C}, \mathsf{D})$ of functors between $\mathsf{C}$ and $\mathsf{D}$. The morphisms between functors are called natural transformations which we introduce now:

**1.11.** Given two functors $F, G : \mathsf{C} \to \mathsf{D}$, a **natural transformation** $\alpha$ is given by: for each object $c \in \mathsf{C}$ a **component** morphism $\alpha_c : F(c) \to G(c)$ in $\mathsf{D}$ such that, for any $f : c \to d$ in $\mathsf{C}$ there is a commutative square as displayed below left.

**natural transformation**

$$\begin{tikzcd}
F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"] & G(c) \arrow[d, "G(f)"] \\
F(d) \arrow[r, "\alpha_d"] & G(d)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(c) \arrow[r, "\alpha_c"'] \arrow[d, "F(f)"] \arrow[rr, bend left, "(\beta\alpha)_c"] & G(c) \arrow[r, "\beta_c"'] \arrow[d, "G(f)"] & H(c) \arrow[d, "H(f)"] \\
F(d) \arrow[r, "\alpha_d"] \arrow[rr, bend right, "(\beta\alpha)_d"'] & G(d) \arrow[r, "\beta_d"] & H(d)
\end{tikzcd}$$

Given two composable natural transformations $\alpha$ and $\beta$ we can obtain a composite by composing the components as displayed above right. With this we obtain, for each pair of categories $\mathsf{C}$ and $\mathsf{D}$, a category $\mathcal{F}\mathrm{un}(\mathsf{C}, \mathsf{D})$ of functors and natural transformations between them. The identity natural transformation $\mathsf{id}_F$ of a functor $F$ has the identity on $F(c)$ for each of it's components, i.e. $(\mathsf{id}_F)_c = \mathsf{id}_{F(c)} : F(c) \to F(c)$.

**1.12.** In line with observation 1.4 that categories often organize models of (essentially) algebraic theories the definition of a natural transformation can be motivated as follows. The domain category $\mathsf{C}$ will serve as an algebraic theory in which every object $c \in \mathsf{C}$ is a sort and the morphisms of $\mathsf{C}$ are function symbols. Then two functors $F, G : \mathsf{C} \to \mathsf{D}$ are thought of as models of $\mathsf{C}$ in the category $\mathsf{D}$. The definition of a natural transformation below is essentially that of a structure preserving map between $\mathsf{C}$ structures.

**1.13.** For any category $\mathsf{C}$ there is the **dual category** $\mathsf{C}^{\mathsf{op}}$ such that $\mathcal{O}\mathrm{bj}(\mathsf{C}^{\mathsf{op}}) = \mathcal{O}\mathrm{bj}(\mathsf{C})$ and

**dual of a category**

$$\mathcal{M}\mathrm{or}(\mathsf{C}^{\mathsf{op}}) := \left\{ f^{\mathsf{op}} : B \to A \mid f : A \to B \text{ in } \mathsf{C} \right\}.$$

Informally the category $\mathsf{C}^{\mathsf{op}}$ is obtained by reversing the arrows of $\mathsf{C}$. That $\mathsf{C}^{\mathsf{op}}$ is even a category is due to the fact that the rules of a category 1.1 are self dual. The usefulness of duality is that each categorical construction or theorem can yield dual notions by passing to dual categories. Often the dual concept is signified by the prefix **co-**.

**1.14.** An functor $F : \mathsf{C} \to \mathsf{D}$ is said to be

(i) **Surjective** if it surjective on objects. This is often weakened to **essentially surjective** which means surjective on isomorphism classes. So for any $d \in \mathsf{D}$ there is an $c \in \mathsf{D}$ such that $F(c) \cong d$.

(ii) **Full** if $F : \mathsf{Hom}(A, B) \to \mathsf{Hom}(F(A), F(B))$ is surjective for each $A, B \in \mathsf{C}$.

(iii) **Faithful** if $F : \mathsf{Hom}(A, B) \to \mathsf{Hom}(F(A), F(B))$ is injective for each $A, B \in \mathsf{C}$.

**subcategory**

An **subcategory** of $\mathsf{C}$ is an category $\mathsf{C}'$ such that some of the objects and morphisms of $\mathsf{C}$ belong to $\mathsf{C}'$. Any subcategory comes with an inclusion functor $i : \mathsf{C}' \to \mathsf{C}$. A **full subcategory** is a subcategory such that the inclusion functor $i$ is full and faithful. This means that if $A$ and $B$ are included in $\mathsf{C}'$ then all morphisms $f : A \to B$ of $\mathsf{C}$ are also in $\mathsf{C}'$

## 1.2 Representables, Yoneda and universal properties

Great utility comes from the fact that $\mathcal{S}\mathrm{et}$ is itself a category while categories (and in particular their hom sets) are defined in terms of $\mathcal{S}\mathrm{et}$. When a category $\mathsf{C}$ is locally small we can study its properties by considering the sets $\mathsf{Hom}(X, Y)$ sitting in $\mathcal{S}\mathrm{et}$ the category of sets. The way the collection of the hom sets $\mathsf{Hom}(X, Y)$ relate to the original category $\mathsf{C}$ is detailed by the Yoneda lemma which we will study in 1.22 and 3. The category of sets is a very nice category in the sense that many constructions can be performed there. Instead of doing a construction in a category $\mathsf{C}$ we can instead work with its representation in $\mathcal{S}\mathrm{et}$ where the construction will most likely make sense.

In this way many questions about our category can be answered. We have the following tools for relating questions about $\mathsf{C}$ to questions about $\mathcal{S}\mathrm{et}$:

1. The relation between a category and its hom sets in $\mathcal{S}\mathrm{et}$ is explained by the fabled Yoneda lemma.
2. If a construction in $\mathcal{S}\mathrm{et}$ on the hom sets descends down to $\mathsf{C}$ we call that construction representable.
3. Finally the adjoint functor theorem will give us a way of deciding when constructions are representable.

**1.15** (Representables)**.** Consider the hom sets $\mathsf{Hom}(A, B)$. Fixing the second argument gives an assignment $A \mapsto \mathsf{Hom}(A, B)$, written $\mathsf{Hom}(-, B)$, from the objects of $\mathsf{C}$ to $\mathcal{S}\mathrm{et}$. Furthermore, given a map $f : A' \to A$ we can precompose any morphism $f : A \to B$ to obtain a morphism $gf : A' \to B$. Stated in terms of hom sets, this gives a function $f^* : \mathsf{Hom}(A, B) \to \mathsf{Hom}(A', B)$. This makes $\mathsf{Hom}(-, B)$ into a functor $\mathsf{C}^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$. Similarly we can fix the first argument to obtain a functor $\mathsf{Hom}(A, -) : \mathsf{C} \to \mathcal{S}\mathrm{et}$, morphisms $f : B \to B'$ are then sent to the postcomposition map $f_* : \mathsf{Hom}(A, B) \to \mathsf{Hom}(A, B')$.

A contravariant functor $F$ natural isomorphic to $\mathsf{Hom}(-, B)$ is called a **representable functor** and $B$ is said to **represent** $F$. Dually, covariant functors $F$ naturally isomorphic to $\mathsf{Hom}(A, -)$ are called **corepresentable functors** although we will also call them **representable**.

**representable functor**

**1.16.** Many categories have interesting (co)representable functors. Examples include

- The contravariant functor $\mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$ sending a topological space $X$ to the set of continuous real valued functions on $X$ is representable. Indeed this functor is equivalent to $\mathsf{Hom}(X, \mathbb{R})$ and so is represented by $\mathbb{R}$.
- The powerset functor $\mathcal{P} : \mathcal{S}\mathrm{et}^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$ sending a set to its set of subsets is representable. Fixing a set $X$ the subsets $A \subset X$ are in bijection with functions $\chi_A : X \to \{0, 1\}$ such that the preimage of $\{1\}$ is $A$. So the powerset functor is represented by $\mathsf{Hom}(-, \{0, 1\})$.
- Given a set (or a topological space) $A$ we can recover the elements of the set (points of the topological space) by looking at $\mathsf{Hom}(1, A)$. So the functor assigning to a set its elements (topological space it's points) is representable by $\mathsf{Hom}(1, -)$.
- The functors sending a category $\mathsf{C}$ to its set of objects $\mathcal{O}\mathrm{bj}(\mathsf{C})$/set of morphisms $\mathcal{M}\mathrm{or}(\mathsf{C})$ are (co)representable by $\mathcal{F}\mathrm{un}([0], \mathsf{C})$ and $\mathcal{F}\mathrm{un}([1], \mathsf{C})$.

**1.17.** Representability is partly useful because it makes defining functors very easy. Compare the normal definition of a functor: assign to each object a set, define the action of morphisms, and finally verify the functor laws; with just picking an object from which functionality automatically follows.

**1.18.** A morphism $f : A \to B$ in a category $\mathsf{C}$ is an

(i) **epimorphism** if precomposition $f^* : \mathsf{Hom}(B, X) \to \mathsf{Hom}(A, X)$ is an injection for all $X \in \mathsf{C}$.

**epimorphism**

(ii) **monomorphism** if postcomposition $f_* : \mathsf{Hom}(X, A) \to \mathsf{Hom}(X, B)$ is an injection for all $X \in \mathsf{C}$.

**monomorphism**

**1.19.** The assignment $B \mapsto \mathsf{Hom}(-, B)$ sends each object of the category $\mathsf{C}$ to a functor $\mathsf{C}^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$. We saw in 1.9 that the collection of such functors is itself a category $\mathcal{F}\mathrm{un}(\mathsf{C}^{\mathsf{op}}, \mathcal{S}\mathrm{et})$. It is then natural to ask whether the assignment above lifts to a functor $\mathbf{y} : \mathsf{C} \to \mathcal{F}\mathrm{un}(\mathsf{C}^{\mathsf{op}}, \mathcal{S}\mathrm{et})$ such that $\mathbf{y}B = \mathsf{Hom}(-, B)$. This is indeed the case: a morphism $f : B \to B'$ yields a natural transformation $f_* : \mathsf{Hom}(-, B) \to \mathsf{Hom}(-, B')$ where the component $(f_*)_A : \mathsf{Hom}(A, B) \to \mathsf{Hom}(A, B')$ is given by postcomposition with $f$. Note that the essential image of the functor $\mathbf{y}$ is the category of representables.

**1.20.** We defined earlier that a functor $F$ is representable if it is naturally isomorphic to $\mathbf{y}B$. In other words, $F$ is representable by $B$ precisely when there are isomorphisms $\mathsf{Hom}(A, B) \to F(A)$ natural in $A$. To determine when a functor $F$ is representable we would first have to decide if there is even a natural transformation $\theta : \mathbf{y}B \to F$. The following characterizes all such natural transformations.

**1.21.** Let us examine, for an arbitrary $F$ and $B$, all possible morphisms $\theta_A : \mathsf{Hom}(A, B) \to F(A)$ natural in $A$. Suppose we had such a natural transformation and for a particular $A$

we knew that the map $f \in \mathsf{Hom}(A,B)$ is sent to $\theta_A(f) = x \in F(A)$ (displayed below right). For any other map $g' \in \mathsf{Hom}(A',B)$ that factors through $f$ as $g' = gf$, the naturality of $\theta$ makes the right hand diagram commute

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
A' \arrow[ur, "g'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, phantom, "\in" description] \arrow[ddd, mapsto, bend right=40] & x \arrow[d, phantom, "\ni" description] \arrow[ddd, mapsto, bend left=40] \\
\mathsf{Hom}(A,B) \arrow[r, "\theta_A"] \arrow[d, "g^*"'] & F(A) \arrow[d, "F(g)"] \\
\mathsf{Hom}(A',B) \arrow[r, "\theta_{A'}"] & F(A') \\
g' \arrow[u, phantom, "\in" description] \arrow[r, mapsto] & F(g)(x) \arrow[u, phantom, "\ni" description]
\end{tikzcd}$$

and so $\theta_{A'}(g') = F(g)(x)$. In other words fixing the assignment $f \mapsto \theta(f)$ already determines how $\theta$ will act on all maps factoring through $f$. But all maps $f$ into $B$ factor through the identity $\mathsf{id}_B \in \mathsf{Hom}(B,B)$ simply as $f = f\mathsf{id}_B$, we must therefore conclude that such natural transformations $\mathbf{y}B \to F$ are wholly determined by the value of $\theta_B(\mathsf{id}_B)$, i.e. an element of $F(B)$. Conversely for any $x \in F(A)$ we get a natural transformation $\theta^x$ generated by sending $\mathsf{id}_B \mapsto x$. We conclude

**1.22.** *Let $\mathsf{C}$ be a category, then for an object $B$ and functor $F : \mathsf{C}^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$ the natural transformations from $\mathbf{y}B$ to $F$, i.e. the set $\mathsf{Nat}(\mathbf{y}B, F)$ is in natural bijection with $F(B)$.*

**Yoneda lemma**

**1.23.** An important corollary of the Yoneda lemma is that the functor $\mathbf{y}$ is fully faithful. This makes the functor $\mathbf{y}$ into an embedding called the **Yoneda embedding**. This is easily seen by setting $F = \mathbf{y}A$ for each $A \in \mathsf{C}$. The Yoneda lemma then states that $\mathsf{Nat}(\mathbf{y}B, \mathbf{y}A) = \mathbf{y}A(B) = \mathsf{Hom}(B,A)$ for each $A$ and $B$ in $\mathsf{C}$.

**Yoneda embedding**

**1.24.** An object $T$ in a category is **terminal** if there is precisely one morphism between any object $X$ and $T$, i.e. every hom set $\mathsf{Hom}(X,T)$ has one element. Equivalently this means that $T$ represents the functor $\mathsf{C}^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$ sending $X \mapsto \{*\}$ to the one element set (then $f \mapsto \mathsf{id}_{\{*\}}$). There is a dual notion of an **initial object** which represents the same, but now covariant, functor.

**terminal object**

**initial object**

**1.25.** Suppose that the functor $F : \mathsf{C}^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$ is represented by an object $C \in \mathsf{C}$ then there is a natural isomorphism $\theta : \mathsf{Hom}(-,C) \cong F(-)$. By the Yoneda lemma such a natural transformation corresponds to an element $x \in F(C)$ which is the image $x = \theta(\mathsf{id}_c)$. This has a converse such that representable objects $C$ for a functor $F$ correspond bijectively with natural isomorphisms $\mathbf{y}C \Rightarrow F$. An object $C \in \mathsf{C}$ has a **universal property** if it represents a functor $F : \mathsf{C}^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$, the choice of object $x \in F(C)$ then specifies a **universal element**.

**universality**

**1.26.** For any functor $F : \mathsf{C}^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$ we can define the **simple Grothendieck construction** $\int_{C \in \mathsf{C}} F(C)$ to be the category with

**simple Grothendieck construction**

(i) As objects it has pair $(C,x)$ with $C \in \mathsf{C}$ and $x \in F(C)$

(ii) An morphism between $(C,x)$ and $(D,y)$ is a map $f : C \to D$ such that $F(f) : y \mapsto x$.

(iii) Identities and composition are simply those from $\mathsf{C}$

This category comes with a canonical functor $\int_{\mathsf{C}} F(-) \to \mathsf{C}$ sending $(C,x) \mapsto C$.

**1.27.** *Let $F$ be a functor, then $C$ is an universal object with universal element $x$ if and only if $(C, x)$ is terminal in the category of elements $\int_{C \in \mathsf{C}} F(C)$*

*Proof.* By the Yoneda lemma an universal pair $(C, x)$ correspond to an natural isomorphism $\theta : \mathsf{Hom}(-, C) \Rightarrow F(-)$ such that $\theta_C : \mathsf{id}_C \mapsto x$. Now suppose that $(D, y) \in \int F$ then we find an unique $f_y = \theta^{-1}(y)$ and this satisfies $F(f_y)(x) = (F(f_y) \circ \theta_C)(\mathsf{id}_C) = \theta_D(f_y) = y$.

Conversely if $(C, x)$ is terminal in $\int F$ then let $\theta : \mathsf{Hom}(-, C) \to F(-)$ be the natural transformation induced by $x$ from the Yoneda lemma. Then for any other $(D, y)$ there is precisely one $f_y$ such that $F(f_y) : x \mapsto y$. But as we let $y$ vary over $F(D)$ this shows that we get that the component $\theta_D : \mathsf{Hom}(D, C) \cong F(D)$ is an bijection. But then $\theta$ is an natural transformation □

## 1.3 Categorical constructions

**1.28.** An **equivalence of categories** between $\mathsf{C}$ and $\mathsf{D}$ is given by functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ with natural isomorphisms $\eta : 1_{\mathsf{C}} \cong GF$ and the $\epsilon : FG \cong 1_{\mathsf{D}}$. Two categories are said to be **equivalent** if there is an equivalence between them.

**equivalence of categories**

**1.29.** The natural transformations $\eta$ and $\epsilon$ can satisfy the following coherence conditions. These conditions are called the triangle laws and make sense for any pair of functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ with $\eta : 1_{\mathsf{C}} \Rightarrow GF$ with $\epsilon : FG \Rightarrow 1_{\mathsf{D}}$. The pair $\eta$ and $\epsilon$ is said to satisfy the **triangle laws** if $G\epsilon \circ \eta G = \mathsf{id}G$ and $\epsilon F \circ F\eta = \mathsf{id}F$, i.e for each $c \in \mathsf{C}$ and $d \in \mathsf{D}$ the following triangle's commute

**triangle laws**

$$\begin{tikzcd}
Fc \arrow[d, "F(\eta_c)"'] \arrow[dr, "\mathsf{id}_{Fc}"] & & Gd \arrow[r, "\eta_{Gd}"] \arrow[dr, "\mathsf{id}_{Gd}"'] & GFGd \arrow[d, "G\epsilon_d"] \\
FGFc \arrow[r, "\epsilon_{Fc}"'] & Fc & & Gd
\end{tikzcd} \tag{1.1}$$

Then the functor $F$ is **left adjoint** to $G$ or $G$ is **right adjoint** to $F$. Together they form an **adjoint pair of functors** or **adjunction** written $F \dashv G$. The natural transformation $\eta$ is called the **unit** of the adjunction and $\epsilon$ is the **counit** of the adjunction.

**left adjoint**
**adjoint pair of functors**

**1.30.** When an equivalence of categories is also an adjunction it is called an **adjoint equivalence**. It is possible to improve an equivalence into an adjoint equivalence by modifying $\epsilon$ to $\epsilon' = \epsilon \circ F(\eta^{-1}) \circ FG(\epsilon^{-1})$ i.e. for each $d \in \mathsf{D}$

**adjoint equivalence**

$$\begin{tikzcd}
 & FGFGd \arrow[r, "F\eta^{-1}_{Gd}"] & FGd \arrow[dr, "\epsilon_d"] & \\
FGd \arrow[ur, "FG\epsilon_d^{-1}"] \arrow[rrr, "\epsilon'_d"] & & & d
\end{tikzcd}$$

This ensures that the laws 1.1 hold. Indeed for one of the laws (the other is similiar) we have, using the naturality of $\eta_{Gd}$:

$$\begin{aligned} G\epsilon'_{Gd}\circ\eta_{Gd} &= G(\epsilon_d)\circ GF(\eta_{Gd}^{-1})\circ GFG(\epsilon_d^{-1})\circ\eta_{Gd} \\ &= G(\epsilon_d)\circ GF(\eta_{Gd}^{-1})\circ\eta_{GFGd}\circ G(\epsilon_d^{-1}) \\ &= G(\epsilon_d)\circ\eta_{Gd}\circ\eta_{Gd}^{-1}\circ G(\epsilon_d^{-1}) \\ &= G(\epsilon_d)\circ G(\epsilon_d^{-1}) \\ &= G(\mathsf{id}_{\mathsf{D}}) = \mathsf{id}_{Gd} \end{aligned}$$

**1.31.** The data of an adjoint pair $F \dashv G$ can be packaged up into a family of isomorphism's $\mathsf{Hom}(Fc,d)\cong\mathsf{Hom}(c,Gd)$ natural in $c\in\mathsf{C}$ and $d\in\mathsf{D}$. This motivates why $F$ is called left-, and $G$ is called right adjoint; they appear on the respective sides in the isomorphism. The isomorphism consist of the horizontal maps displayed below.

$$\begin{tikzcd} & \mathsf{Hom}(GFc,Gd) \arrow[dr, "-\circ\eta_c"] & \\ \mathsf{Hom}(Fc,d) \arrow[ur, "G"] \arrow[rr, shift left, "G(-)\circ\eta_c"] & & \mathsf{Hom}(c,Gd) \arrow[ll, shift left, "\epsilon_d\circ F(-)"] \arrow[dl, "F"] \\ & \mathsf{Hom}(Fc,FGd) \arrow[ul, "\epsilon_d\circ -"] & \end{tikzcd}$$

The triangle laws 1.1 and the naturality of $\eta$ and $\epsilon$ show that the mappings are inverse.

The natural isomorphism $\theta(c,d):\mathsf{Hom}(Fc,d)\cong\mathsf{Hom}(c,Gd)$ shows that $Fc$ represents the functor $d\mapsto\mathsf{Hom}(c,Gd)$ and that $Gd$ represents the functor $c\mapsto\mathsf{Hom}(Fc,d)$. The Yoneda lemma states that the natural isomorphisms going either way are determined by the image of the identity. Indeed the identity under the isomorphism becomes the unit $\eta_c=\theta(c,Fc)(\mathsf{id}_{Fc})$ and counit $\epsilon_d=\theta(d,Gd)^{-1}(\mathsf{id}_{Gd})$ of the adjunction. This shows that the 'natural isomorphism' definition is equivalent to the unit-counit definition of an adjunction.

**1.32.** For any pair of categories $\mathsf{D}$ and $\mathsf{C}$ and an object $c\in\mathsf{C}$ the constant functor $\Delta(c):\mathsf{D}\to\mathsf{C}$ is the functor such that $\Delta(c)(d)=c$ for all $d\in\mathsf{D}$. Furthermore if we have a map $f:c\to c'$ in $\mathsf{C}$ we get a natural transformation $\overline{f}:\Delta(c)\Rightarrow\Delta(c')$ where each component is just $f$. This makes $\Delta$ into a functor from $\mathsf{C}\to\mathcal{F}\mathrm{un}(\mathsf{D},\mathsf{C})$.

**cone**

**1.33.** Let $p:\mathsf{D}\to\mathsf{C}$ be any functor then a **cone** over $p$ is an object $c\in\mathsf{C}$ with a natural transformation $\mu:\Delta(c)\Rightarrow p$. Concretely this means for any $d\in\mathsf{D}$ a map $\mu_d:c\to p(d)$ such that for any $f:d\to d'$ the following triangle commutes

$$\begin{tikzcd} & c \arrow[dl, "\mu_d"'] \arrow[dr, "\mu_{d'}"] & \\ p(d) \arrow[rr, "p(f)"] & & p(d') \end{tikzcd}$$

**1.34.** Given a functor $p:\mathsf{D}\to\mathsf{C}$ we can form the functor $\mathsf{Cone}(p):\mathsf{C}^{\mathsf{op}}\to\mathcal{S}\mathrm{et}$ assigning to each object $c\in\mathsf{C}$ the set $\mathsf{Nat}(\Delta(c),p)$ of cones of $p$ over $c$. This is a functor since a map $f:d\to c$ turns a cone over $c$ into a cone over $d$ by precomposition. If the functor

$\mathsf{Cone}$ is representable we call the representing object the **limit** of the functor $p$ and write it as $\mathsf{lim}(p)$. When the limit exists representability means that there is an isomorphism $\mathsf{Hom}(c, \mathsf{lim}(p)) \cong \mathsf{Nat}(\Delta(c), p)$ natural in $c$. By the Yoneda lemma, a natural transformation out of $\mathsf{Hom}(-, \mathsf{lim}(p))$ is fully determined by the image of $\mathsf{id}_{\mathsf{lim}(p)}$. We call the corresponding cone $\lambda : \Delta(\mathsf{lim}(p)) \Rightarrow p$ the **limiting cone**. Then any map $f : c \to \mathsf{lim}(p)$ yields a cone $f\lambda$ over $c$ with legs $(\lambda f)_d = \mu_d \circ f$. Because $\mathsf{lim}(p)$ represents the $\mathsf{Cone}$ functor the above assignment is an bijection and so we get an inverse: Every cone $\mu : \Delta(c) \Rightarrow p$ yields a unique map $c \to \mathsf{lim}(p)$ displayed below such that all the triangles involving $\mu_d, \lambda_d$ and the induced map, commute.

**limit**

$$\begin{tikzcd}
 & & p(d) \arrow[dd, "p(f)"] \\
c \arrow[r, dashed] \arrow[urr, bend left, "\mu_d"] \arrow[drr, bend right, "\mu_{d'}"'] & \mathsf{lim}(p) \arrow[ur, "\lambda_d"'] \arrow[dr, "\lambda_{d'}"] & \\
 & & p(d')
\end{tikzcd}$$

**1.35.** The above discussion dualizes as follows, for any functor $p : \mathsf{D} \to \mathsf{C}$:

| | | |
|---|---|---|
| Functor | cone functor $\mathsf{Cone}(p)$<br>$c \mapsto \mathsf{Nat}(\Delta(c), p)$ | cocone functor $\mathsf{coCone}(p)$<br>$c \mapsto \mathsf{Nat}(p, \Delta(c))$ |
| Representing object | The limit $\mathsf{lim}(p)$ | The colimit $\mathsf{colim}(p)$ |
| Natural isomorphism | $\mathsf{Hom}(c, \mathsf{lim}(p)) \cong \mathsf{Nat}(\Delta(c), p)$ | $\mathsf{Hom}(\mathsf{lim}(p), c) \cong \mathsf{Nat}(p, \Delta(c))$ |
| Universal element | the limiting cone<br>$\lambda : \Delta(\mathsf{lim}(p)) \Rightarrow p$ | the colimiting cocone<br>$\lambda : p \Rightarrow \Delta(\mathsf{lim}(p))$ |

**1.36.** The functor $p : \mathsf{D} \to \mathsf{C}$ for which we find a (co)limit is called a **diagram** and the category $\mathsf{D}$ is called the **shape of the diagram**. Important examples of (co)limits in a category $\mathsf{C}$ of a specific shape are

**diagram**

- For shape $\mathsf{Disc}(0)$, the empty category, there is a unique diagram $\mathsf{Disc}(0) \to \mathsf{C}$. The limit $\mathbf{1}$ over this diagram is called the **initial object** and the colimit $\varnothing$ is called the **terminal object**.
- For shape $\mathsf{Disc}(2)$, the discrete category with two objects, a diagram $\mathsf{Disc}(2) \to \mathsf{C}$ picks out two objects $A$ and $B$ in $\mathsf{C}$. The limit $A \times B$ is called the **binary product** and the colimit $A + B$ is called the **binary coproduct**.
- For any cardinal $\kappa$ a diagram $\mathsf{Disc}(\kappa) \to \mathsf{C}$ picks out an object $X_i \in \mathsf{C}$ for each $i < \kappa$. The limit $\prod_{i<\kappa} X_i$ is called the product, similarly the colimit $\sum_{i<\kappa} X_i$ is called the coproduct. Letting $\kappa = 0$ and $\kappa = 2$ yield the previous examples.
- For the shape $\mathsf{D}_{\|} = \{\cdot \rightrightarrows \cdot\}$ a diagram $\mathsf{D}_{\|} \to \mathsf{C}$ picks out objects $A, B \in \mathsf{C}$ with two parallel morphisms $f : A \to B$ and $g : A \to B$. The limit $\mathsf{eq}(f, g)$ is called the **equalizer** and the colimit $\mathsf{coeq}(f, g)$ the **coequalizer** of $f$ and $g$.
- For shape $\mathsf{D}_{\mathsf{cospan}}$ displayed below a diagram $\mathsf{D}_{\mathsf{cospan}} \to \mathsf{C}$ picks out two morphisms $f : A \to B$ and $p : E \to B$ with common codomain. The limit $A \times_B E$ ($f$ and $p$ are implicit) is called the pullback of $f$ and $p$.

$$\mathsf{D}_{\mathsf{cospan}} = \left\{ \begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \right\} \qquad \mathsf{D}_{\mathsf{span}} = \left\{ \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \end{tikzcd} \right\}$$

- Dually, for shape $\mathsf{D}_{\mathsf{span}} = \mathsf{D}_{\mathsf{cospan}}^{\mathsf{op}}$ displayed above a diagram $\mathsf{D}_{\mathsf{span}} \to \mathsf{C}$ picks out two morphisms $f : A \to X$ and $g : A \to Y$ in $\mathsf{C}$ with common domain. The colimit $X +_A Y$ ($f$ and $g$ are implicit) is called the pushout of $f$ and $g$.

Let $\lambda$ be a cardinal, a category $\mathsf{D}$ is called **$\lambda$-filtered** if for each $\lambda$-small diagram, i.e. diagram with less than $\lambda$ arrows, has a cocone. For example $\mathsf{D}$ is $\omega$-filtered, also called finitely filtered, if

**filtered category**

- for each $a, b \in \mathsf{D}$ there is an object $c$ such that $a \to c$ and $b \to c$;
- and for each parallel pair of morphisms $f, g : a \to b$ there is an morphism $e : b \to c$ such that $ef = eg$.

A colimit of an $\lambda$-filtered diagram is called a $\lambda$-filtered colimit.

**1.37.** If in a category $\mathsf{C}$, all diagrams of shape $\mathsf{D}$ have limits resp. colimits then the category $\mathsf{C}$ is said to have all $\mathsf{D}$-limits resp. $\mathsf{D}$-colimits. This leads to notions such as: **categories having products, coproducts, equalizers, coequalizers, pullbacks, pushouts, filtered colimits, etc**.

**categories having colimits/limits**

A category $\mathsf{C}$ has products iff $\mathsf{C}^{\mathsf{op}}$ has coproducts. Similarly for initial and terminal; equalizers and coequalizers; pullbacks and pushouts.

**1.38** ((co)limits as adjunction)**.** There is a category $\mathcal{F}\mathrm{un}(\mathsf{D}, \mathsf{C})$ of all diagrams of a certain shape. If a category $\mathsf{C}$ has all $\mathsf{D}$-limits then the family of isomorphisms $\mathsf{Hom}(c, \mathsf{lim}(p)) \cong \mathsf{Nat}(\Delta(c), p)$ is also natural in $p \in \mathcal{F}\mathrm{un}(\mathsf{D}, \mathsf{C})$. Naturality in both $p$ and $c$ means that there is an adjunction $\Delta \dashv \mathsf{lim}$. Dually if a category $\mathsf{C}$ has all $\mathsf{D}$-colimits we have $\mathsf{colim} \dashv \Delta$.

**1.39.** We end our chapter with some observation about the existence of (co)limits. Every observation about limits comes with a dual observation about colimits indicated in parenthesis.

A category $\mathsf{C}$ has

- binary products if it has pullbacks and an initial object. The product of $A$ and $B$ is then given by $A \times_{\mathbf{1}} B$ the pullback over $A \to \mathbf{1} \leftarrow B$. Dually, it has binary coproducts if it has pushouts and an initial object, where the coproduct of $A$ and $B$ is given by $A +_{\varnothing} B$.
- **finite (co)products** if the following equivalent properties hold
  (i) $\mathsf{C}$ has $n$-(co)products for all $n \in \mathsf{Nat}$
  (ii) $\mathsf{C}$ has an initial (terminal) object and binary (co)products
  *Proof.* Take the (co)product 'one by one', i.e. $\prod X_i = ((\mathbf{1} \times X_0) \times X_1) \times ...$ $\square$

**finite (co)products**

- **small (co)products** if for every set $S$ it has $|S|$-(co)products.

small (co)products

- **finite (co)limits** if the following equivalent properties hold
  - (i) For any shape $\mathsf{D}$ such that $\mathcal{M}\mathrm{or}(D)$ is finite it has all $\mathsf{D}$-(co)limits.
  - (ii) It has binary (co)products and (co)equalizers.

finite (co)limits

- **all small (co)limits** if it has $\mathsf{D}$-(co)limits for all small categories $\mathsf{D}$.

all small (co)limits

**1.40.** Let $F : \mathsf{C} \to \mathsf{D}$ be a functor and $p : I \to \mathsf{C}$ be any diagram in $\mathsf{C}$.

- If $\mathsf{lim}(p)$ exists, then $F$ **preserves** this limit if $F(\mathsf{lim}(p)) = \mathsf{lim}(F \circ p)$.
- If $X \in \mathsf{C}$ and $F(X) = \mathsf{lim}(F \circ p)$ then $F$ **reflects** this limit if $X = \mathsf{lim}(p)$.
- If $\mathsf{lim}(F \circ p)$ exists then $F$ **creates** or **lifts** this limits if there is an $X \in \mathsf{C}$ such that $F(X) = \mathsf{lim}(F \circ p)$ and moreover $X = \mathsf{lim}(F)$.

Similarly we can talk about functors preserving/reflecting/creating certain colimits, or all limits of a certain shape, or all limits in general.

**1.41.** Let $\mathsf{C}$ and $\mathsf{D}$ be categories, then there is a **product category** $\mathsf{C} \times \mathsf{D}$. The product category has as objects pairs of objects $(c, d) \in \mathcal{O}\mathrm{bj}(\mathsf{C} \times \mathsf{D})$ with $c \in \mathcal{O}\mathrm{bj}(\mathsf{C})$ and $d \in \mathcal{O}\mathrm{bj}(\mathsf{D})$. A morphism $(f, g)$ from $(c, d)$ to $(c', d')$ is a pair of morphisms $f : c \to c'$ in $\mathsf{C}$ and $g : d \to d'$ in $\mathsf{D}$.

product category

**1.42.** For any category $\mathsf{C}$ we obtain the functor $\mathsf{C} \times - : \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$, this functor has a right adjoint $\mathcal{F}\mathrm{un}(-, \mathsf{C})$ assigning to a category $\mathsf{E}$ the category of functors from $\mathsf{C}$ to $\mathsf{E}$. This makes the subcategory of small categories cartesian closed with the functor categories serving as exponential objects.

**1.43.** The product category is the product in the category of small categories. The above makes the category of small categories an cartesian closed category.

Recall that the counit of such an adjunction is called evaluation. At $\mathcal{S}\mathrm{et} \in \mathcal{C}\mathrm{at}$ it is given by $\mathsf{ev} : \mathsf{C} \times \mathcal{F}\mathrm{un}(\mathcal{S}\mathrm{et}, \mathsf{C}) \to \mathcal{S}\mathrm{et}$ on objects given by $\mathsf{ev}(c, P) = P(c)$. We will write $\mathsf{ev}_c : \mathcal{P}\mathrm{sh}(\mathsf{C}) \to \mathcal{S}\mathrm{et}$ given by $\mathsf{ev}_c(P) = \mathsf{ev}(c, P)$, this makes $\mathsf{ev}_{(-)}$ into functor.

**1.44.** A natural transformation $\theta : F \Rightarrow G$ between $F, G : \mathsf{C} \to \mathsf{D}$ is represented by a functor $\mathsf{C} \times \mathbf{2} \to \mathsf{D}$.

## 1.4 Monads

**1.45.** A **monad** on a category $\mathsf{C}$ is an endofunctor $T : \mathsf{C} \to \mathsf{C}$ together with **unit** $\eta : \mathsf{id}_{\mathsf{C}} \to T$ and **multiplication** $\mu : T^2 \to T$ natural transformations such that the following diagrams commute

monad

$$\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\
 & T &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}$$

Similarly a **comonad** on D is an endofunctor with a **counit** $\epsilon : T \Rightarrow \mathsf{id}_D$ and **comultiplication** $\mu : T \Rightarrow T^2$.

**comonad**

**1.46.** Every adjunction $F : \mathsf{D} \rightleftarrows \mathsf{C} : U$ induces a monad on C given by $T = FU$, the unit of the adjunction $\eta : \mathsf{id}_C \Rightarrow FU$ serves as unit for the monad, and the multiplication is given by $\mu = F\epsilon U$. The triangle laws confirm that this is a monad. Dually, this adjunction also induces a comonad on D.

The mapping from adjunctions to monads on C or comonads on D has an section: given a monad $T : \mathsf{C} \to \mathsf{C}$ there is a category $\mathsf{C}^T$ called the category of $T$-algebras with an adjunction $F : \mathsf{C}^T \rightleftarrows \mathsf{C} : U$.

**1.47.** An $T$-**algebra** is an object $A \in \mathsf{C}$ together with a map f: $TA \to A$ such that the following diagrams commute

**algebra for a monad**

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "\mathsf{id}_A"'] & TA \arrow[d, "f"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, "\mu_A"] \arrow[d, "Tf"'] & TA \arrow[d, "f"] \\
TA \arrow[r, "f"] & A
\end{tikzcd}$$

A **morphism of** $T$-**algebras** between $f : TA \to A$ and $g : TB \to B$ is a map $k : A \to B$ interacting well with the monad structure in the sense the the following diagram commutes

$$\begin{tikzcd}
TA \arrow[r, "Tk"] \arrow[d, "f"'] & TB \arrow[d, "g"] \\
A \arrow[r, "k"] & B
\end{tikzcd}$$

This gives a category $\mathsf{C}^T$ of $T$-algebras.

**1.48.** There is an obvious forgetful functor $U^T : \mathsf{C}^T \to \mathsf{C}$ sending an $T$-algebra $f : TA \to A$ to $A$. This functor has an left adjoint $F^T : \mathsf{C} \to \mathsf{C}^T$ sending an object $A \in \mathsf{C}$ to the **free** $T$-**algebra** on $A$. This is the object $TA$ with a map $\mu : T^2A \to TA$ as $T$-algebra map.

**free algebra**

**1.49.** In general there can be many adjunctions $F \dashv U$ with $U : \mathsf{D} \to \mathsf{C}$ inducing the same monad $T$ on C. In fact we can consider the category of such adjunctions inducing the same monad on C. The category of $T$-algebras is then terminal in this category, i.e for each adjunction as above there is a canonical map $\mathsf{D} \to \mathsf{C}^T$ sending $d \in \mathsf{D}$ to $F^T(U(d))$.

**1.50.** An adjunction natural isomorphic to the adjunction between C and a category $\mathsf{C}^T$ of $T$-algebras is called a **monadic adjunction**. Similarly a functor $U : \mathsf{D} \to \mathsf{C}$ is **monadic** if it fits in a monadic adjunction.

**monadic adjunction**

**1.51.** *A monadic functor* $U : \mathsf{D} \to \mathsf{C}$ *creates all limits.*

*Proof.* Let $T$ be the induced monad, we can then take $\mathsf{D} \cong \mathsf{C}^T$ by assumption. Since $U$ is a right adjoint it preserves all limits.

Let $p : \mathsf{I} \to \mathsf{C}^T$ be a diagram such that $\mathsf{lim}(U \circ p)$ exists. Write $p(i) : TA_i \to A_i$, then $U \circ p(i) = A_i$ such that $\mathsf{lim}_i(A_i) = \mathsf{lim}(U \circ p)$. For $i \to j$ in I we can draw a piece of the diagram $p$ as the back side of

$$\begin{tikzcd}
TA_i \arrow[rr] \arrow[dd, "p(i)"'] & & TA_j \arrow[dd, "p(j)"] \\
& T(\mathrm{lim}_i\, A_i) \arrow[ul, "T(\pi_i)"] \arrow[ur, "T(\pi_j)"'] \arrow[dd, dashed, "f"] & \\
A_i \arrow[rr] & & A_j \\
& \mathrm{lim}_i\, A_i \arrow[ul, "\pi_i"] \arrow[ur, "\pi_j"'] &
\end{tikzcd}$$

Now $T(\mathrm{lim}_i\, A_i)$ is already a cone over $\{TA_i\}_{i\in I}$ and also forms a cone over $U \circ p$. Since $\mathrm{lim}_i\, A_i$ is the limit we find the unique dashed map $f$ displayed above. We claim that $f$ is the limit of $p$ in $\mathsf{C}^T$.

To show that $f$ is an $T$-algebra we have to show that $f\eta_{\mathrm{lim}_i\, A_i} = \mathrm{id}_{\mathrm{lim}_i\, A_i}$. Since these are maps into a limit is is enough to show that both define the same cone over $U \circ p$. The outer side of the diagram below are the $i$th legs of the cones. The right hand square commutes by assumption, the left hand square commutes since $\eta$ is natural, and the bottom triangle commutes since $p(i)$ is an $T$-algebra.

$$\begin{tikzcd}
\mathrm{lim}_i\, A_i \arrow[r, "\eta_{\mathrm{lim}_i\, A_i}"] \arrow[d, "\pi_i"] & T(\mathrm{lim}_i\, A_i) \arrow[r, "f"] \arrow[d, "T(\pi_i)"] & A \arrow[d, "\pi_i"] \\
A_i \arrow[r, "\eta_{A_i}"] \arrow[rr, bend right, "\mathrm{id}_{A_i}"'] & T(A_i) \arrow[r, "p(i)"] & A_i
\end{tikzcd}$$

$$\begin{tikzcd}
T^2A \arrow[rrr, "\mu_A"] \arrow[dr, "T^2(\pi_i)"] \arrow[ddd, "T(f)"'] & & & TA \arrow[dl, "T(\pi_i)"] \arrow[ddd, "f"] \\
& T^2A_i \arrow[r, "T(\mu_{A_i})"] \arrow[d, "T(p(i))"] & TA_i \arrow[d, "p(i)"] & \\
& TA_i \arrow[r, "p(i)"] & A_i & \\
TA \arrow[rrr, "f"] \arrow[ur, "T(\pi_i)"] & & & A \arrow[ul, "\pi_i"']
\end{tikzcd}$$

A similar argument validates the other law displayed above right. We want to show that $\pi_i f \mu_A = \pi_i f T(f)$. In the picture the inner square commutes since $p(i)$ is an $T$-algebra, the other squares commute by assumption. □

**1.52.** *We say that a monad is* ***idempotent*** *if any of the following equivalent conditions are fulfilled* **idempotent monad**

- *(i) The multiplication $\mu : T^2 \Rightarrow T$ is an natural isomorphism.*
- *(ii) The multiplication $\mu : T^2 \Rightarrow T$ is a component wise monomorphism.*
- *(iii) The natural transformations $T\eta$ and $\eta T$ from $T \Rightarrow T^2$ are equal.*
- *(iv) Any $T$-algebra map $TA \to A$ is an isomorphism.*

*Proof.* (i) ⇒ (ii). Is trivial.
(ii) ⇒ (iii). The monad laws assert that $\mu\eta T = \mu T\eta$, since $\mu$ is mono this means $\eta T = T\eta$.
(iii) ⇒ (iv). For any $T$-algebra $\eta_A f = T(f)\eta_{TA}$ by naturality of $\eta$. By (iii) we have $\eta_{TA} = T(\eta_A)$ and so $\eta_A f = T(f\eta_A)$, but $f\eta_A = \mathrm{id}_A$ and so $\eta_A$

$$\begin{tikzcd}
TA \arrow[r, "\eta_{TA}=T(\eta_A)"] \arrow[d, "f"] \arrow[dr, dashed, sloped, "T(f\eta_A)=\mathrm{id}"] & T^2A \arrow[d, "Tf"] \\
A \arrow[r, "\eta_A"] & TA
\end{tikzcd}$$

(iv) ⇒ (i). Every component $\mu_A$ is in particular an $T$-algebra map and so an isomorphism.

$\square$

**1.53.** Suppose that $i : \mathsf{D} \to \mathsf{C}$ is an fully faithful right adjoint, then the induced monad $T : \mathsf{C} \to \mathsf{C}$ is idempotent. Moreover if an object $A$ admits an $T$-algebra $f$ then point (iv) above shows that this map is an isomorphism, because any such $T$-algebra has right inverse $\eta_A$ this makes this $T$-algebra unique. Finally in such a situation the functor $i$ is actually monadic [Rie17, 5.3.3].

# Chapter 2

# Reflective subcategories, localization and factorization systems

## 2.1 Reflective subcategories

Suppose we have a nice category $\mathcal{E}$ with finite limits and all colimits. Often $\mathcal{E}$ will be that category of set based models of an algebraic theory. Examples include the categories of groups or the category of presheaves on a site $X$. We will be concerned with a full subcategory $\mathsf{C} \hookrightarrow \mathcal{E}$, this corresponds to those objects/models satisfying some property. Tautologically, this is the property of 'belonging to $\mathsf{C}$', in practice the property will define the subcategory.

**2.1.** Examples of this situation are

- $\mathsf{Ab} \hookrightarrow \mathcal{G}\mathrm{rp}$, groups which are abelian
- $\mathcal{S}\mathrm{h}(X) \hookrightarrow \mathcal{P}\mathrm{sh}(X)$, presheaves which are sheaves with respect to the open covers of $X$
- $\mathcal{K}\mathrm{an} \hookrightarrow \mathcal{P}\mathrm{sh}(\Delta)$, simplicial sets satisfying the Kan condition

The embeddings above are forgetful functors in the sense that they forget the property defining the subcategory in question. In this light, it is natural to wonder if the inclusion admits a left adjoint $L : \mathcal{E} \to \mathsf{C}$ which corresponds to freely forcing the property to hold.

**2.2.** A full subcategory $\mathsf{C} \hookrightarrow \mathcal{E}$ such that the inclusion has a left adjoint $L : \mathsf{C} \to \mathcal{E}$ is called a **reflective subcategory**. Analogously, a functor $L : \mathcal{E} \to \mathsf{C}$ with a full and faithful right adjoint is called the **reflector**. The reflector produces from $e \in \mathcal{E}$ an object $L(e) \in \mathsf{C}$ which we can think of as an approximation of $e$ satisfying the property defining the subcategory.

**reflective subcategory**

**2.3.** For the full subcategories above we get

- $\begin{tikzcd} \mathsf{Ab} \arrow[r, hook, bend left] \arrow[r, phantom, "\top"] & \mathcal{G}\mathrm{rp} \arrow[l, bend left] \end{tikzcd}$, to any group $G$ we can associate its abelianization which is the quotient $G/[G,G]$ of $G$ by the commutator subgroup $[G,G]$.
- $\begin{tikzcd} \mathcal{S}\mathrm{h}(X) \arrow[r, hook, bend left] \arrow[r, phantom, "\top"] & \mathcal{P}\mathrm{sh}(\mathsf{X}) \arrow[l, bend left] \end{tikzcd}$, any presheaf $P$ has a sheafification $\mathbf{a}P$ defined in 3.34.
- Non example: the inclusion $\mathcal{K}\mathrm{an} \hookrightarrow \mathcal{P}\mathrm{sh}(\Delta)$ does not have a left adjoint. There is however the fibrant replacement functor $R : \mathcal{P}\mathrm{sh}(\Delta) \to \mathcal{K}\mathrm{an}$ sending an simplicial set $S$ to it's fibrant replacement which is almost left adjoint to the inclusion (see 6.2).

$$\begin{tikzcd} \mathsf{C} \arrow[r, hook, bend left, "i"] \arrow[r, phantom, "\top"] & \mathcal{E} \arrow[l, bend left, "L"] \end{tikzcd}$$

Lets consider the general situation $i \dashv L$ as displayed above. The endofunctor $Li : \mathsf{C} \to \mathsf{C}$, takes an object $c \in \mathsf{C}$, forgets the property of belonging to $\mathsf{C}$ and then freely makes it into an $\mathsf{C}$ object. Intuitively, we would expect that $Lic \cong c$, this is in fact the case by isomorphisms $\epsilon_c : Lic \to c$: these are natural in $c$ and form the counit $\epsilon : Li \to \mathsf{id}_{\mathsf{C}}$ as the following lemma shows.

**2.4.** *Suppose we have an adjunction $L \dashv i$ with counit $\epsilon$, then*

*(i) i is faithful $\iff$ $\epsilon$ is a component wise epimorphism*

*(ii) i is full $\iff$ $\epsilon$ is a component wise split monomorphism*

*(iii) i is full and faithful $\iff$ $\epsilon$ is a component wise isomorphism*

*Proof.* Note that (i) and (ii) $\implies$ (iii). For (i) and (ii) we need the following observation. In terms of the unit and counit the isomorphism of the adjunction $\mathsf{Hom}(Lc, d) \to \mathsf{Hom}(c, id)$ sends $f \mapsto if \circ \eta_{ic}$, for any $c \in \mathsf{C}$ and $d \in \mathsf{D}$. After precomposition with $\epsilon_c$ we obtain

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & f \circ \epsilon_c \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & i(f \circ \epsilon_c) \circ \eta_{ic} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ \mathsf{Hom}(c, d) \arrow[r, "\epsilon_c^*"] \arrow[rr, dashed, bend right, "i(-)"'] & \mathsf{Hom}(Lic, d) \arrow[r, "\mathsf{adj}"] & \mathsf{Hom}(ic, id) \end{tikzcd}$$

By functoriality and the triangle law: $i(f \circ \epsilon_c) \circ \eta_{ic} = if \circ i\epsilon_c \circ \eta_{ic} = if$, so the composite is just the action of the functor $i$ on hom sets. Recall that $i$ is faithful resp full iff the action on hom sets is injective resp surjective. Since $\mathsf{adj}$ as displayed above is an isomorphism, the $i$ action on hom sets is injective resp surjective iff $\epsilon_c^*$ is injective resp surjective. But then by [Rie17, 1.2] this is the case iff $\epsilon_c$ is epi resp. split mono. ☐

**2.5.**
- Theorem 2.4 has a dual version relating the full/faithfullness of the left adjoint to analogous properties of $\eta$. Instead of precomposition with the counit, postcompose with the unit and use the hom isomorphism of the adjunction in the other direction.

- The above also highlights why fibrant replacement $R : \mathcal{P}\mathrm{sh}(\Delta) \to \mathcal{K}\mathrm{an}$ can never be part of an reflective adjunction. Indeed take the kan complex $\Delta^0$, this complex has one simplex in each dimension (degenerate for dimensions greater than 0). Fibrant replacement glues at least one extra 1 non degenerate simplex onto $\Delta^0$ so there can be no isomorphism between $\Delta^0$ and $R(\Delta^0)$.

**2.6** (limits and colimits)**.** Reflectivity of a subcategory $i : \mathsf{C} \hookrightarrow \mathcal{E}$ allows easy computation of limits and colimits in $\mathsf{C}$ that $\mathcal{E}$ admits. For example, if we have a diagram $p : \mathsf{D} \to \mathsf{C}$ such that the induced diagram $ip : \mathsf{D} \to \mathcal{E}$ has a limit $e = \mathsf{lim}(ip) \in \mathcal{E}$ then $e \in \mathsf{C}$ and forms the limit of $p$. This is because $i$ is monadic, see 1.53, and so creates all limits, 1.51. We can already show how colimits are computed which we now show

**2.7.** *Suppose there is a reflective subcategory* $i : \mathsf{C} \hookrightarrow \mathcal{E}$ *with reflector* $L : \mathcal{E} \to \mathsf{C}$*. The colimit of a diagram* $p : \mathsf{D} \to \mathsf{C}$ *exists when* $ip : \mathsf{C} \to \mathcal{E}$ *admits a colimit* $\mathsf{colim}(ip)$ *and is then given by the reflection* $L(\mathsf{colim}(ip))$*.*

*Proof.* Suppose $l = \mathsf{colim}(ip) \in \mathcal{E}$ exists and let $\lambda : ip \Rightarrow \Delta(l)$ be its colimiting cone. Since left adjoints preserve limits, we get a colimiting cone $L\lambda : Lip \to \Delta(Ll)$ over the diagram $Lip$ in $\mathsf{C}$. But $\epsilon : Li \cong \mathsf{id}_{\mathsf{C}}$ and so the diagram $Lip$ is equivalent to $p$, hence $Ll$ is a colimit of $p$. □

**2.8.** The adjoint pair $i \dashv L$ induces a comonad $Li$ on $\mathsf{C}$ and monad $iL$ on $\mathcal{E}$. The counit shows that the comonad is equivalent to the identity. More interesting is that monad $iL : \mathcal{E} \to \mathcal{E}$, here the counit shows that $iLiL \cong i\mathsf{id}_{\mathsf{C}}L \cong iL$ and so the functor $iL$ is idempotent. The pair $i, L$ is then a splitting of the idempotent $iL$ in $\mathsf{C}$, indeed $L$ is epi and $i$ is mono and even full and faithful. This allows us to consider $L$ as a coequalizer of $iL$ and $\mathsf{id}_{\mathcal{E}}$.

**2.9.** There is a dual notion of a **coreflective** subcategory, the properties are simple dualizations of the above discussion which we will now summarize:

**coreflective**

- A subcategory $i : \mathsf{C} \hookrightarrow \mathcal{E}$ is coreflective if the inclusion has a right adjoint $R : \mathcal{E} \to \mathsf{C}$ called the **coreflection**.
- Colimits of a coreflective subcategory are closed under colimits and so are computed as if in $\mathcal{E}$. A limits can be be computed in $\mathcal{E}$ and then coreflected into $\mathsf{C}$.
- Since $i$ is full and faithful the unit $\eta$ of its right adjoint $R$ is an isomorphisms. Therefore the monad $Ri$ is the identity and the comonad $iR$ is an idempotent.

## 2.2 Localization

**2.10.** Just as for groups, it will turn out that a (co)reflective subcategory is essentially determined by it's 'kernel'. The kernel of a functor $L : \mathcal{E} \to \mathsf{C}$ will be the preimage of $\mathsf{Core}(\mathsf{C})$ the **core of a category** $\mathsf{C}$, the subcategory of all isomorphisms. This makes $\mathsf{Core}(\mathsf{C})$ into the maximal subgroupoid of $\mathsf{C}$. The reason for not picking the subcategory of say the identity maps, which might seem to be a more natural choice, is because this is not invariant under equivalence of categories. The **kernel of an functor** is $\mathsf{ker}(L) = L^{-1}(\mathsf{Core}(\mathsf{C}))$, i.e. the category of morphisms in $\mathcal{E}$ that get sent to isomorphisms in $\mathsf{C}$.

**core of a category**

**kernel of an functor**

**2.11.** Let $\mathsf{Sub}_{\mathsf{full}}(\mathcal{E})$ denote the poset of full subcategories of $\mathcal{E}$. Inside this poset we find subposets of both the reflective and coreflective subcategories. Similarly let $\mathsf{Sub}_{\mathsf{wide}}(\mathcal{E})$ be the poset of wide subcategories containing all isomorphisms. This notation is a bit non standard, normally a wide subcategory is a category containing all identities, but this is not invariant under equivalence of categories. With this we can describe the kernel as a contravariant functor $\mathsf{ker} : \mathsf{Sub}_{\mathsf{full}}(\mathcal{E}) \to \mathsf{Sub}_{\mathsf{wide}}(\mathcal{E})$.

**2.12.** Conversely we can see if we can generate a reflective or coreflective subcategory from an wide subcategory $\mathsf{S}$, it will then be more convenient to refer to the collection of arrows $S = \mathcal{M}\mathrm{or}(\mathsf{S})$. Suppose we have a class of maps $S$ of $\mathcal{E}$ then

- An object $X \in \mathcal{E}$ is $S$**-local** if for each $f : A \to B$ in $S$ the induced $\mathsf{Hom}(f, X) : \mathsf{Hom}(B, X) \to \mathsf{Hom}(A, X)$ is an isomorphism. $S$**-local**
- An map $f : A \to B$ in $\mathcal{E}$ is an **(left)** $S$**-equivalence** if for any $S$-local object $X$ the induced $\mathsf{Hom}(f, X) : \mathsf{Hom}(B, X) \to \mathsf{Hom}(A, X)$ is an isomorphism. **(left)** $S$**-equivalence**
- An object $X \in \mathcal{E}$ is $S$**-colocal** or $S$**-resolvant** if the induced $\mathsf{Hom}(X, f) : \mathsf{Hom}(X, A) \to \mathsf{Hom}(X, B)$ is an isomorphism. $S$**-colocal**
- An map $f : A \to B$ in $\mathcal{E}$ is **(right)** $S$**-equivalence** if for any $S$-colocal object $X$ the induced $\mathsf{Hom}(X, f)$ is an isomorphism. **(right)** $S$**-equivalence**

We now give some properties of left $S$-equivalences, dual properties hold for right $S$-equivalences.

(i) If $f$ is an left (right) $S$-equivalences then any right (left) inverse $g$ is also an left (right) $S$-equivalence. Indeed we have for any $S$-local object

$$\begin{tikzcd}
\mathsf{Hom}(A, X) \arrow[r, "f^*"] \arrow[rr, bend left, "(fg)^* = \mathsf{id}^*"] & \mathsf{Hom}(B, X) \arrow[r, "g^*"] & \mathsf{Hom}(A, X)
\end{tikzcd}$$

where $f^*$ and $\mathsf{id}^*$ are isomorphisms by assumption, so $g^*$ must be an isomorphism as well.

(ii) An $S$-equivalence between $S$-local objects is an isomorphism. The definition of an $S$-equivalence yields an bijection $f^* : \mathsf{Hom}(B, A) \to \mathsf{Hom}(A, A)$. Then the preimage of $\mathsf{id}_B$ yields an right inverse $g : B \to A$. By the previous point $g$ is also an $S$-equivalence an so we obtain a further right inverse $h$ of $g$. But then $f = h$ and this is two sided inverse of $g$, showing that $f$ is an isomorphisms.

(iii) Every $f \in S$ is an $S$-equivalence, this follows easily.

**2.13.** *If $S = \mathsf{ker}(L)$ for some reflector $L : \mathcal{E} \to \mathsf{C}$, then*

*(i) the category $\mathsf{C}$ is equivalent to the full subcategory spanned by the $S$-local objects*

*(ii) and the left $S$-equivalences are just $S$.*

*Dually if $S = \mathsf{ker}(R)$ for a coreflector $R : \mathcal{E} \to \mathsf{C}$, then $\mathsf{C}$ is the category of $S$-colocal objects and right $S$-equivalences are $S$.*

*Proof.* The equivalence of categories will be witnessed by $i$. Since it is already full and faithful we only have to show that its essential image is the category of $S$-local objects.

We first show that for any $X \in \mathsf{C}$ the image $iX$ is $S$-local. For any $f : A \to B$ in $S$ we obtain using the adjunction the following commutative square

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, mapsto] \arrow[dr, phantom, sloped, "\ni"] & & & xf \arrow[ddd, mapsto] \arrow[dl, phantom, sloped, "\in"] \\
& \mathsf{Hom}(B, iX) \arrow[r, "f^*"] \arrow[d, phantom, sloped, "\cong"] & \mathsf{Hom}(A, iX) \arrow[d, phantom, sloped, "\cong"] & \\
& \mathsf{Hom}(LB, X) \arrow[r, "(Lf)^*"'] \arrow[dl, phantom, sloped, "\ni"] & \mathsf{Hom}(LA, X) \arrow[dr, phantom, sloped, "\in"] & \\
\epsilon_X Lx \arrow[rrr, mapsto] & & & \epsilon_x LxLf
\end{tikzcd}$$

but $Lf$ is an isomorphism, so $(Lf)^*$ is as well which means $f^*$ is too.

Now for surjectivity, suppose that $X$ is $S$-local then $LX \in \mathsf{C}$ satisfies $x \cong iLX$. Moreover this is given by the unit map $\eta_X : X \to iLX$. By assumption this is a map between $S$-local objects, so it will be enough to show that $\eta_X$ is an $S$-equivalence. The map $L\eta_X$ has, by the triangle laws, a left inverse $\epsilon_{iX}$, but the counit components are isomorphisms so its left inverse $L\eta_X$ is also an isomorphism. This shows that $\eta_X \in \mathsf{ker}(L)$ hence $\eta_X \in S$ which means in particular that $\eta_X$ is an $S$-equivalence.

(ii) if $f$ is any $S$-equivalence then for any $S$-local object $X$ we have $\mathsf{Hom}(B, X) \cong \mathsf{Hom}(B, iLX)$ by the unit and so we can apply the result from above showing that

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[dddd, mapsto] \arrow[dr, phantom, sloped, "\ni"] & & & xf \arrow[dddd, mapsto] \arrow[dl, phantom, sloped, "\in"] \\
& \mathsf{Hom}(B, X) \arrow[r, "f^*"] \arrow[d, phantom, sloped, "\cong"] & \mathsf{Hom}(A, X) \arrow[d, phantom, sloped, "\cong"] & \\
& \mathsf{Hom}(B, iLX) \arrow[r, "f^*"] \arrow[d, phantom, sloped, "\cong"] & \mathsf{Hom}(A, iLX) \arrow[d, phantom, sloped, "\cong"] & \\
& \mathsf{Hom}(LB, X) \arrow[r, "(Lf)^*"'] \arrow[dl, phantom, sloped, "\ni"] & \mathsf{Hom}(LA, X) \arrow[dr, phantom, sloped, "\in"] & \\
\epsilon_X Lx \arrow[rrr, mapsto] & & & \epsilon_x LxLf
\end{tikzcd}$$

then the same square shows that $Lf$ has to be an isomorphism and so $f \in S$. ☐

**2.14.** This shows that the kernel map is split mono: we can reconstruct the reflective localization given the maps $S$ sent to isomorphisms as the category of $S$-local objects. This localization has the property that all maps in $S$ become isomorphisms on $S$-local objects. On the other hand it is not the case that any collection of morphisms are the kernel of a reflective localization, or any functor for that matter. Indeed if $f$ is a map with a right inverse $g$ and $g \in \mathsf{ker}(L)$ for some $L$ then also $f \in \mathsf{ker}(L)$. We do have the following partial inverse.

**2.15.** Given a collection of morphisms $W \subset \mathcal{M}\mathrm{or}(\mathcal{E})$ to invert we can form the **localization of $\mathcal{E}$ at** $W$ which is $\mathcal{E}[W^{-1}]$ together with an identity on objects functor $\gamma : \mathcal{E} \to \mathcal{E}[W^{-1}]$. Intuitively we obtain $\mathcal{E}[W^{-1}]$ by formally inverting the morphisms in $W$.

**localization**

The category $\mathcal{E}[W^{-1}]$ is a category with the same objects as $\mathcal{E}$ and whose morphisms from $A$ to $B$ are **zig-zag chains** of morphisms alternatingly form $\mathcal{E}$ and $W^{-1}$.

$$A \xrightarrow{\in\mathcal{E}} \cdot \xleftarrow{\in W} \cdot \xrightarrow{\in\mathcal{E}} \cdot \xleftarrow{\in W} \cdots \xleftarrow{\in W} \cdot \xrightarrow{\in\mathcal{E}} B$$

Quotiented by the equivalence relation where

- Composable maps can be composed and we can freely add identity maps
- We can remove any $\cdot \xleftarrow{f} \cdot \xrightarrow{f} \cdot$.

For the exact definition see [Rie19].

**2.16.** The most important thing about this definition is not the construction itself but the universal property it enjoys.

**2.17.** *For any functor $F : \mathcal{E} \to \mathsf{C}$ such that $F(f)$ is an isomorphism if $f \in W$, factors uniquely through $\gamma$ as displayed below.*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "\gamma"] & \mathsf{C} \\
\mathcal{E}[W^{-1}] \arrow[ur, dotted, "F_!"] &
\end{tikzcd}$$

*Proof.* By assumption for every $f \in W$ there is an unique map $i_f \in \mathsf{C}$ inverse to $F(f)$. Now $F_!$ acts the same on objects as $F$. On zig-zag chains it sends all forward morphisms to the image under $F$, and sending backward morphisms $f$ to the forward morphism $i_f$. This map is clearly unique and it defines an chain of composable morphisms which has a unique composite in $\mathsf{C}$. Moreover it is clear that $F_! \circ \gamma = F$. ☐

## 2.3 Decomposition

**2.18.** Suppose we have a category $\mathsf{C}$, we will examine the situation in which we are able to decompose this category into two simpler wide subcategories $\mathsf{L} \hookrightarrow \mathsf{C} \hookleftarrow \mathsf{R}$ such that the smallest subcategory of $\mathsf{C}$ containing $\mathsf{L}$ and $\mathsf{R}$ is the whole of $\mathsf{C}$. We will write $l_0, l_1, ...$ and $r_0, r_1, ...$ to indicate that maps belong to $\mathsf{L}$ and $\mathsf{R}$ respectively.

In general a morphism $f$ of $\mathsf{C}$ will decompose into an arbitrary long chain $r_1 \circ l_1 \circ r_2 \circ \cdots$. These unwieldy chains are undesirable so we will restrict our attention to such decomposition where we may distribute left maps over right maps in the following sense. Given a composeable pair $l$ and $r$ there are $l'$ and $r'$ such that $lr = r'l'$. Note that $\mathsf{id}$ is both in $\mathsf{L}$ and $\mathsf{R}$ due to both being wide subcategories. In particular we can always write $r\mathsf{id} = \mathsf{id}r$ and similarly for $l$.

**2.19.** There is a decomposition system on $\mathcal{S}\mathsf{et}$ where $\mathsf{L}$ is the category of surjective functions and $\mathsf{R}$ is the category of injective functions.

With such a decomposition system we can easily retrieve $\mathsf{C}$ from the subcategories $\mathsf{L}$ and $\mathsf{R}$ as follows.

**2.20.** Let $\mathsf{LR}$ be the category with the same objects as $\mathsf{C}$ and whose morphisms are composable pairs $(l, r)$ with $l \in \mathsf{L}$ and $r \in \mathsf{R}$. The composition uses the distributor: $(l, r) \circ (l'', r'') = (l \circ l', r' \circ r)$ where $r'l' = l''r$. The identity at $C$ is given by $(\mathsf{id}, \mathsf{id})$ which is an identity by the law of the distributor.

**2.21.** *We have a full and surjective functor* $U : \mathsf{LR} \to \mathsf{C}$*.*

*Proof.* We define the map $U : \mathsf{LR} \to \mathsf{C}$ which is the identity on objects and sends an arrow $U(l, r) = r \circ l$. This functor is obviously surjective. Moreover since $\mathsf{C}$ is a subcategory of the category freely generated by $\mathsf{L}$ and $\mathsf{R}$ we can write any arrow $f = rl$ for some $l$ and $r$, now clearly $U(r, l) = f$ so $f$ is full. ☐

**2.22.** We might ask for the functor to be faithful but it turns out this is way too strong. For example the surjective-injective decomposition system of 2.19 would not yield an faithful functor even though decompositions are unique up to unique isomorphism. Instead we might ask for the functor to be 'essentially faithful' where we define a notion of isomorphism between maps of $\mathsf{LR}$. Even though such a notion is easy enough to define the whole approach is rather ad-hoc. Instead we will replace the category $\mathsf{LR}$ to one with an natural notion of isomorphism.

**2.23.** Recall that the ordinal category $\mathbf{2}$ is the category consisting of two objects $0, 1$ and a single non identity arrow $0 \to 1$. For any category $\mathsf{C}$ we can then define $\mathsf{C}^{\mathbf{2}}$ to be the functor category $\mathcal{F}\mathrm{un}(\mathbf{2}, \mathsf{C})$ called the **category of arrows** of $\mathsf{C}$. Concretely, the objects of $\mathsf{C}^{\mathbf{2}}$ are morphisms $f, g$ of $\mathsf{C}$ and an arrows $u : f \to g$ in $\mathsf{C}^{\mathbf{2}}$ is a commutative square

**category of arrows**

$$\begin{tikzcd} \cdot \arrow[r, "u_0"] \arrow[d, "f"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "u_1"] & \cdot \end{tikzcd}$$

**2.24.** The subcategories $\mathsf{L}$ and $\mathsf{R}$ induce full subcategories of $\mathsf{C}^{\mathbf{2}}$ which we will write as $\mathsf{C}^{\mathbf{2}}|_{\mathsf{L}}$ and $\mathsf{C}^{\mathbf{2}}|_{\mathsf{R}}$. Then $f$ is an object of $\mathsf{C}^{\mathbf{2}}|_{\mathsf{L}}$ when $f$ is an arrow of $\mathsf{L}$, but we retain all morphisms from $\mathsf{C}^{\mathbf{2}}$, i.e. commutative squares in $\mathsf{C}$. We can then take the pullback of the categories $\mathsf{C}^{\mathbf{2}}|_{\mathsf{L}}$ and $\mathsf{C}^{\mathbf{2}}|_{\mathsf{R}}$ along the codomain and domain projections displayed below left.

$$\begin{tikzcd} \mathsf{C}^{\mathbf{2}}|_{\mathsf{L}} \times_{\mathsf{C}} \mathsf{C}^{\mathbf{2}}|_{\mathsf{R}} \arrow[r] \arrow[d] & \mathsf{C}^{\mathbf{2}}|_{\mathsf{R}} \arrow[d, "\mathsf{dom}"] \\ \mathsf{C}^{\mathbf{2}}|_{\mathsf{L}} \arrow[r, "\mathsf{cod}"] & \mathsf{C} \end{tikzcd}$$

Composing the projections $\pi_{\mathsf{L}}, \pi_{\mathsf{R}}$ of $\mathsf{C}^{\mathbf{2}}|_{\mathsf{L}} \times_{\mathsf{C}} \mathsf{C}^{\mathbf{2}}|_{\mathsf{R}}$ with the inclusions $i_{\mathsf{L}} : \mathsf{C}^{\mathbf{2}}|_{\mathsf{L}} \hookrightarrow \mathsf{C}^{\mathbf{2}}$ and $i_{\mathsf{R}} : \mathsf{C}^{\mathbf{2}}|_{\mathsf{R}} \hookrightarrow \mathsf{C}^{\mathbf{2}}$ we obtain, by the universal property of the pullback, a canonical monomorphism $i = \langle i_{\mathsf{L}}\pi_{\mathsf{L}}, i_{\mathsf{R}}\pi_{\mathsf{R}} \rangle$. To see that this map is mono, suppose we have two morphisms $x, y$ such that $ix = iy$. For $\mathsf{L}$ we get $i_{\mathsf{L}}\pi_{\mathsf{L}}x = i_{\mathsf{L}}\pi_{\mathsf{L}}y$ so $\pi_{\mathsf{L}}x = \pi_{\mathsf{L}}y$ since $i_{\mathsf{L}}$ is mono. Repeating the same for $\mathsf{R}$ we find that $x$ and $y$ induce two identical limiting cones over the pullback diagram so by the universal property $x = y$.

$$\begin{tikzcd}
\mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}} \arrow[r, "\sim"] \arrow[d, hook] & \mathsf{C}^2|_{\mathsf{L}} \times_{\mathsf{C}} \mathsf{C}^2|_{\mathsf{R}} \arrow[d, hook, "i"] \\
\mathsf{C}^3 \arrow[r, "\langle s_2, s_0 \rangle"] & \mathsf{C}^2 \times_{\mathsf{C}} \mathsf{C}^2
\end{tikzcd}$$

Since $\mathsf{C}$ is a category, the map $\langle s_2, s_0 \rangle : \mathsf{C}^3 \to \mathsf{C}^2 \times_{\mathsf{C}} \mathsf{C}^2$ sending a commutative triangle $r \circ l = f$ to the composable arrows $r \circ l$ is an isomorphism, this essentially states that compositions exists uniquely. Now by the pullback stability of monomorphisms and isomorphisms the above pullback square shows that $\mathsf{C}^2|_{\mathsf{L}} \times_{\mathsf{C}} \mathsf{C}^2|_{\mathsf{R}}$ is isomorphic to a subcategory of $\mathsf{C}^3$ which we will henceforth denote $\mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}}$.

With these preliminaries we can now package up the data of a decomposition system $(\mathsf{L}, \mathsf{R})$ of a category $\mathsf{C}$ into $\mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}}$ equipped with the composition map $p$ to $\mathsf{C}^2$. The statement that every arrow $f$ of $\mathsf{C}$ decomposes into some $r \circ l$ is equivalent to the statement that $p$ is surjective on objects. The essential uniqueness of decompositions is then equivalent to $p$ being fully faithful which we will see in the next section.

**2.25.** Suppose we have a decomposition system $(\mathsf{L}, \mathsf{R})$ on $\mathsf{C}$, by the above remarks this correspond to an surjective functor $p : \mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}} \to \mathsf{C}^2$. It is then natural to consider a situation where this functor has a section $d : \mathsf{C}^2 \to \mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}}$. In fact, such any such map $d : \mathsf{C}^2 \to \mathsf{C}^3$ determines such a decomposition system where $\mathsf{L}$ is the image of $s_2 \circ d$ and $\mathsf{R}$ is the image of $s_0 \circ d$ in $\mathsf{C}$.

**2.26.** An **functorial decomposition system** is a functor $d : \mathsf{C}^2 \to \mathsf{C}^3$.

**functorial decomposition system**

## 2.4 Factorization and lifting properties

**2.27.** In relation with decompositions of a category one often encounters lifting properties of maps in the left class against maps of the right class. A map $l$ has the left lifting property with respect to $r$, or $r$ has the right lifting property with respect to $l$, if for each diagram commutative square of the shape

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook, "l"] & E \arrow[d, two heads, "r"] \\
Y \arrow[r, "i"] \arrow[ur, dotted] & B
\end{tikzcd}$$

there exists a unique diagonal filler $i$ making everything commute. We will write $l \boxslash r$ in this situation. If in addition the filler is unique we say that the map $l$ is left orthogonal to $r$ and we write $l \perp r$. Analogously a class of maps $\mathcal{L}$ is said to have left lifts against (resp. be orthogonal to) a class $\mathcal{R}$ if for each $l \in \mathcal{L}$ and each $r \in \mathcal{R}$ we have $l \boxslash r$ (resp $l \perp r$), in this case we write $\mathcal{L} \boxslash \mathcal{R}$ (resp. $\mathcal{L} \perp \mathcal{R}$).

**2.28.** *We prove the 'if' directions first. Suppose we have two subcategories* $\mathsf{L}$ *and* $\mathsf{R}$ *of* $\mathsf{C}$. *Then* $\mathsf{L} \boxslash \mathsf{R}$ *if and only if* $p : \mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}} \to \mathsf{C}^2$ *is full. And* $\mathsf{L} \perp \mathsf{R}$ *if and only if* $p : \mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}} \to \mathsf{C}^2$ *is full and faithful.*

*Proof.* Suppose we have two decompositions $f = rl$ and $f' = r'l'$ in $\mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}}$ and an arrow between $(u, v) : f \to f'$ in $\mathsf{C}^2$, this means that the diagram without $w$ below left commutes. Now rearrange this diagram to the one displayed below right, now if $\mathsf{L} \perp \mathsf{R}$ we find the filler $w$. But then $w$ is also the desired lift in the diagram on the left so $p$ is full. Now if there were another lift $w'$ in the diagram on the right, i.e. $p$ is not faithful, then this would also be an diagonal filler on the right contradicting the uniqueness implied by $\mathsf{L} \perp \mathsf{R}$.

$$\begin{tikzcd}
\cdot \arrow[r, "l"'] \arrow[rr, bend left, "f"] \arrow[d, "u"] & \cdot \arrow[r, "r"'] \arrow[d, dashed, "w"] & \cdot \arrow[d, "v"] \\
\cdot \arrow[r, "l'"] \arrow[rr, bend right, "f'"'] & \cdot \arrow[r, "r'"] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "l'u"] \arrow[d, "l"'] & \cdot \arrow[d, "r'"] \\
\cdot \arrow[r, "vr"'] \arrow[ur, dashed, "w"] & \cdot
\end{tikzcd}$$

Now for the 'only if' directions. Given a lifting problem of $l$ against $r$ displayed below left. We can extend it to an arrow of $\mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}}$ as displayed below right since identities are part of $\mathsf{L}$ and $\mathsf{R}$. Then the existence of a $w$ on the right gives the required lift on the left. Suppose we have another lift $w'$ in the diagram on the right, then this also fits into the diagram on the left. Now if $p$ is faithful then $w = w'$ and so lifts are unique.

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "l"'] & \cdot \arrow[d, "r"] \\
\cdot \arrow[r, "v"'] \arrow[ur, dashed, "w"] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "l"'] \arrow[rr, bend left, "l"] \arrow[d, "u"'] & \cdot \arrow[r, "\mathrm{id}"'] \arrow[d, dashed, "w"] & \cdot \arrow[d, "v"] \\
\cdot \arrow[r, "\mathrm{id}"] \arrow[rr, bend right, "r"'] & \cdot \arrow[r, "r"] & \cdot
\end{tikzcd}$$

□

Now if the map $p : \mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}} \to \mathsf{C}^2$ is essentially surjective on objects we get that for any $f$ in $\mathsf{C}$ we find a decomposition $rl$ such that $rl$ and $f$ are isomorphic as objects of $\mathsf{C}^2$. In particular we have isomorphisms $u$ and $v$ such that the outer rectangle below commutes.

$$\begin{tikzcd}
\cdot \arrow[r, "l"] \arrow[d, "u"'] & \cdot \arrow[r, "r"] \arrow[dr, "vr"] & \cdot \arrow[d, "v"] \\
\cdot \arrow[ur, "lu^{-1}"'] \arrow[rr, "f"'] & & \cdot
\end{tikzcd}$$

Now if we stipulate that every isomorphism of $\mathsf{C}$ belongs to $\mathsf{R}$ and $\mathsf{L}$ we can follow that $p$ is in fact surjective on objects by noting that $lu^{-1}$ belongs to $\mathsf{L}$ and $vr$ belongs to $\mathsf{R}$ above. Now among other things the orthogonality of $\mathsf{L}$ and $\mathsf{R}$ imply that decompositions are unique up to unique isomorphism. This is because two decompositions $rl$ and $r'l'$ of the same map fit into two lifting diagrams giving unique arrows both ways. What is more surprising is that canonically unique decompositions also imply that the left class is orthogonal to the right class. The following is due to Joyal.

**2.29.** *Given a decomposition structure* $(\mathsf{L}, \mathsf{R})$ *on* $\mathsf{C}$*, then decompositions are unique up to unique isomorphism if and only if* $\mathsf{L} \perp \mathsf{R}$

*Proof.* The 'only if' side follows from the argument given above. For the 'if' part we first show uniqueness. Given a lifting problem $(u, v)$ of $l$ against $r$ as displayed on the left below. Factor $u$ and $v$ as displayed. Then the dotted arrows are composites $l_v l$ and $r r_u$ and so in $\mathsf{L}$ and $\mathsf{R}$ respectively. Thus we have two $(\mathsf{L}, \mathsf{R})$ decompositions of the map going from $S$ to $T'$ and hence obtain a unique isomorphism $k$. Then $r_u w l_v$ is a lift of $l$ against $r$.

$$\begin{tikzcd}
S \arrow[r, "l_u"] \arrow[d, "l"'] \arrow[dr, dotted, "\in\mathsf{L}"] \arrow[rr, bend left, "u"] & \cdot \arrow[r, "r_u"] \arrow[dr, dotted, "\in\mathsf{R}"] & S' \arrow[d, "r"] \\
T \arrow[r, "l_v"'] \arrow[rr, bend right, "v"'] & \cdot \arrow[r, "r_v"'] \arrow[u, dashed, "k"] & T'
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "l_u"] \arrow[dd, "l"'] \arrow[dr, dotted, "\in\mathsf{L}"] \arrow[rr, bend left, "u"] & \cdot \arrow[r, "r_u"] & S' \arrow[dd, "r"] \\
 & \cdot \arrow[u, dashed, "i"] \arrow[ur, "r_f"] \arrow[dr, dotted, "\in\mathsf{R}"] & \\
T \arrow[r, "l_v"'] \arrow[ur, "l_f"] \arrow[rr, bend right, "v"'] & \cdot \arrow[r, "r_v"'] \arrow[u, dashed, "j"] & T'
\end{tikzcd}$$

Now suppose that there is another lift $f$ which we will immediately decompose as $r_f l_f$ as displayed above right. Then the dotted composites are again in $\mathsf{L}$ and $\mathsf{R}$ respectively so we get double decomposition's of the arrows $u$ and $v$ and so obtain unique isomorphisms $i$ and $j$. Now $ij$ is again an isomorphism and moreover $ijl_v l = l_u$ and $rr_u ij = r_v$ and so $k = ij$ by the uniqueness of the isomorphism. But then $f = r_f l_f = r_u ij l_v = r_u k l_v$ which was the diagonal lift we found previously. We conclude that $\mathsf{L} \perp \mathsf{R}$. □

**2.30.** An easy way to obtain orthogonal or weakly orthogonal classes of maps in a category $\mathsf{C}$ is to start with some class of maps and take all maps having the relevant liftings against this class. Starting with a class $\mathcal{I}$ or $\mathcal{J}$ of $\mathcal{M}\mathrm{or}(\mathsf{C})$ let

$$\mathcal{I}^{\boxslash} := \{r \in \mathsf{C} : \mathcal{I} \boxslash r\}, \qquad {}^{\boxslash}\mathcal{J} := \{l \in \mathsf{C} : l \boxslash \mathcal{J}.\}$$

. Then $\mathcal{J} \boxslash {}^{\boxslash}\mathcal{J}$ and $\mathcal{I}^{\boxslash} \boxslash \mathcal{I}$. Similarly we have $(-)^{\perp}$ and ${}^{\perp}(-)$ producing orthogonal classes.

**2.31.** The operations $(-)^{\boxslash}$ and ${}^{\boxslash}(-)$ are functors $\mathcal{PM}\mathrm{or}(\mathsf{C}) \to \mathcal{PM}\mathrm{or}(\mathsf{C})$ on the powerset of morphisms of a category $\mathsf{C}$, recall that the powerset is a poset category under inclusion. They actually form an contravariant self adjunction in the sense that

$$\mathcal{J} \subset \mathcal{I}^{\boxslash} \iff \mathcal{I} \subset {}^{\boxslash}\mathcal{J}$$

such an contravariant adjunction between posets is called an **antitone Galois connection** and it produces a closure operation on the right, and a closure operation on the left. These are obtained by composing $(-)^{\boxslash}$ and ${}^{\boxslash}(-)$ and correspond to the monad and comonad formed by iterating the functors of the adjunction:

$$c\ell_{\mathsf{left}}(\mathcal{I}) := {}^{\boxslash}(\mathcal{I}^{\boxslash}), \quad c\ell_{\mathsf{right}}(\mathcal{J}) := ({}^{\boxslash}\mathcal{J})^{\boxslash}$$

These satisfy $\mathcal{I} \subset c\ell_{\mathsf{left}}(\mathcal{I})$ and $c\ell_{\mathsf{left}}(c\ell_{\mathsf{left}}(\mathcal{I})) = c\ell_{\mathsf{left}}(\mathcal{I})$, the same holds for $c\ell_{\mathsf{right}}$.

**(weak) factorization system**

**2.32.** A **(weak) factorization system** is a decomposition $(\mathsf{L}, \mathsf{R})$ such that equivalently $\mathsf{L} = {}^{\boxslash}\mathsf{R}$ or $\mathsf{L}^{\boxslash} = \mathsf{R}$. We say that the is a **functorial factorization system** if it comes with a section $\mathsf{C}^2 \to \mathsf{C}^3|^{\mathsf{L}}_{\mathsf{R}}$.

**orthogonal factorization system**

Correspondingly an **orthogonal factorization system** or **strict factorization system** is defined analogously with $\perp$ replacing $\boxslash$. Orthogonal factorization systems are automatically functorial because the decompositions are unique.

**2.33.** We say that $\mathcal{I}$ generates a weak (resp. orthogonal) factorization system from the left if $(c\ell_{\mathsf{left}}(\mathcal{I}), \mathcal{I}^{\boxslash})$ is a weak (resp. orthogonal) factorization system. Dually the class $\mathcal{J}$ generates a weak (resp. orthogonal) factorization system from the right if $({}^{\boxslash}\mathcal{J}, c\ell_{\mathsf{right}}(\mathcal{J})$ is a weak (resp. orthogonal) factorization system.

For Topoi, which we consider later, we will need a certain lifting property against monomorphisms in a category. For this we introduce the following

**2.34.** Let $m : S \rightarrowtail T$ be an monomorphism in $\mathsf{C}$. Consider an object $E \in \mathsf{C}$ and then any map $v : S \to E$, we want to consider the possible diagonal fillers completing the diagram below.

$$\begin{tikzcd} S \arrow[r, "v"] \arrow[d, tail, "m"] & E \\ T \arrow[ur, dotted] & \end{tikzcd} \qquad \mathsf{Hom}(T, E) \xrightarrow{m^*} \mathsf{Hom}(S, E)$$

Equivalently we can consider the induced precomposition map $m^*$ between the hom sets as shown to the right. An object $E$ is

- $f$**-separated** if fillers are unique when they exist, in other words $m^*$ is injective. **$f$-separated**
- $f$**-fibrant** if fillers exists, in other words $m^*$ is surjective. **$f$-fibrant**
- an $f$**-sheaf** if fillers exist uniquely, in other words $m^*$ is bijective. **$f$-sheaf**

Note that if an object is both $f$-separated and $f$-fibrant it is an $f$-sheaf.

We can relativize the above, replacing the object $E$ with a map $p : E \to B$. We then require an additional map $u : A \to B$ replacing the lifting triangle with a lifting square

$$\begin{tikzcd} S \arrow[r, "v"] \arrow[d, tail, "m"] & E \arrow[d, "p"] \\ T \arrow[ur, dotted] \arrow[r, "u"] & B \end{tikzcd} \qquad \mathsf{Hom}(T, E) \xrightarrow{(m^*, p_*)} \mathsf{Hom}(S, E) \times_{\mathsf{Hom}(S,B)} \mathsf{Hom}(T, B)$$

For the induced hom map corresponding to the lifting problem now sends a diagonal from $m$ to $p$ to the maps obtained by pre- and postcompostition with $p$ and $m$ respectively. The pullback on the right are precisely pairs of maps like $v$ and $u$ such that further composition with $m$ and $p$ yield the same map from $S$ to $B$, i.e. such that the square commutes.

An map $p : E \to B$ is

- $f$**-separating** if fillers are unique when they exist, in other words $(m^*, p_*)$ is injective. **$f$-separating**
- $f$**-fibration** if fillers exists, in other words $(m^*, p_*)$ is surjective. **$f$-fibration**

- an $f$-**orthogonal** if fillers exist uniquely, in other words $(m^*, p_*)$ is bijective.

**$f$-orthogonal**

To digest this definition we have the following theorem.

**2.35.** *In a category* $\mathsf{C}$ *with finite limits if a map* $p$ *is* $f$*-seperating (fibration, orthogonal) then all of its fibres are* $f$*-seperated (fibrant, sheaf).*

*Proof.* Indeed suppose that we want to extend $v$ along $m$ then we get a lifting problem of $m$ against $p$ which by assumption admits the dotted diagonal filler $f$ of the back square. By the universal property of the pullback we get a unique map $(f, !)$ that is the extension of $f$ along $v$. Any other extension clearly induces another lift of $m$ against $p$ so they must equal $(f, !)$.

$$\begin{tikzcd}
S \arrow[rr] \arrow[dr, "v"] \arrow[dd, tail, "m"'] & & E \arrow[dd, "p"] \\
& t^*E \arrow[ur, "\tilde{t}"'] \arrow[dd] & \\
T \arrow[rr] \arrow[dr, "!"'] \arrow[ur, dashed, "(f,!)"'] \arrow[uurr, dotted, bend left, "f"] & & B \\
& 1 \arrow[ur, "t"'] &
\end{tikzcd}$$

□

**2.36.** We can extend the above definition to classes of maps $\mathcal{J}$, requiring that an $\mathcal{J}$-fibration (resp $\mathcal{J}$-separated, $\mathcal{J}$-sheaf) if it is an $f$-fibrant (resp $f$-separated, $f$-sheaf) for all $f \in \mathcal{J}$. This is in fact an instance of extension by colimits which we will meet in 3.9, because $\mathcal{J}$ is simply the union (which is a colimit) of the singletons $\{j\}$.

**2.37.** We wish to already emphasize already the following parallel between parts to come:

**section 3.31** Let $J$ be an Grothendieck topology on $\mathsf{C}$. This is equivalent to a collection of monomorphisms $\mathcal{J}$ of the category $\mathcal{P}\mathrm{sh}(\mathsf{C})$ 3.31. Then the sheaves with respect to this topology are precisely the $\mathcal{J}$-sheaves in the above sense.

**section 5.33** Let $\mathcal{J}$ be a class of generating cofibrations (resp anodyne maps) of a cofibrantly generated model category $\mathsf{C}$. Then the trivial fibrations (resp. fibrations) of $\mathsf{C}$ are precisely the $J$-fibrations.

# Chapter 3

# Topos theory

**3.1.** For a small category $\mathsf{C}$ a contravariant functor $F : \mathsf{C}^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$ will be called a **presheaf** on $\mathsf{C}$. The collection of preasheaves on a small category $\mathsf{C}$ form the category $\mathcal{P}\mathrm{sh}(\mathsf{C})$ which is the **category of presheaves on** $\mathsf{C}$. This is just the functor category $\mathcal{F}\mathrm{un}(\mathsf{C}^{\mathsf{op}}, \mathcal{S}\mathrm{et})$ meaning that a morphism of preshaeves $F \to G$ is a natural transformation between $F$ and $G$.

**presheaf**

$\mathcal{P}\mathbf{sh}(\mathsf{C})$

**3.2.** The Yoneda embedding shows that the there is a full and faithful functor $\mathbf{y} : \mathsf{C} \to \mathcal{P}\mathrm{sh}(\mathsf{C})$. This allows us to consider $\mathcal{P}\mathrm{sh}(\mathsf{C})$ as a kind of extension of $\mathsf{C}$. The nature of this extension will be elucidated in 3.1.

**3.3.** *The category $\mathcal{P}\mathit{sh}(\mathsf{C})$ has all small limits and colimits and they are computed pointwise in the sense that for a diagram $X_\bullet : \mathsf{I} \to \mathcal{P}\mathit{sh}(\mathsf{C})$*

$$(\lim_i X_i)(c) := \lim_i X_i(c), \quad (\operatorname{colim}_i X_i)(c) := \operatorname{colim}_i X_i(c),$$

*Proof.* Since $\mathcal{S}\mathrm{et}$ has all small limits and colimits the above functors are well defined elements of the $\mathcal{P}\mathrm{sh}C$. To verify that it is actually the limit (the colimit proof works the same) we verify the universal property

$$\mathsf{Hom}(Y, \lim_i X_i) \cong \mathsf{Nat}(\Delta Y, X_\bullet)$$

A natural transformation $\tau \in \mathsf{Nat}(\Delta Y, X_\bullet)$ is represented by a functor $\overline{\tau} : \mathcal{F}\mathrm{un}(\mathsf{I} \times \mathbf{2}, \mathcal{P}\mathrm{sh}(\mathsf{C}))$ such that

$$\overline{\tau}(i,0) = Y, \quad \overline{\tau}(i, 0 \to 1) = \tau_i, \quad \overline{\tau}(i,1) = X_i$$

By applying the cartesian closed structure on $\mathcal{C}\mathrm{at}$ we can transform this to a functor $F : \mathsf{C}^{\mathsf{op}} \to \mathcal{F}\mathrm{un}(\mathsf{I} \times \mathbf{2}, \mathcal{S}\mathrm{et})$ such that

$$F(c) : (i,0) \mapsto Y(c), \quad F(c) : (i, 0 \to 1) \mapsto \tau_i(c), \quad F(c) : (i,0) \mapsto X_i(c),$$

But for any $c$ this is just a natural transformation $F(c) \in \mathsf{Nat}(\Delta(Y(c)), X_\bullet(c))$ which by the universal property of the limit corresponds to $f_c \in \mathsf{Hom}(Y(c), \lim_i X_i(c))$. This yields an unique family of maps $f_c : Y(c) \to \mathsf{lim}_i X_i(c)$ which is moreover natural in $c$ by the naturality of $\tau$. So we get the natural transformation $f : Y \to \mathsf{lim}_i X_i$ as required. $\square$

**3.4.** The category of presheaves $\mathcal{P}\mathrm{sh}(\mathsf{C})$ inherits many more pleasant properties from $\mathcal{S}\mathrm{et}$. This, together with the remark that $\mathsf{C}$ is canonically embedded into $\mathcal{P}\mathrm{sh}(\mathsf{C})$ by the Yoneda embedding $\mathbf{y}$, makes $\mathcal{P}\mathrm{sh}(\mathsf{C})$ a very good context for studying $\mathsf{C}$: If $\mathsf{C}$ misses a property that makes a certain construction impossible then we can attempt it on its image in $\mathcal{P}\mathrm{sh}(\mathsf{C})$ under the Yoneda embedding. Consequently we don't need to ask 'Does $\mathsf{C}$ admit a certain construction?' but instead can ask 'Does the construction in $\mathcal{P}\mathrm{sh}(\mathsf{C})$ descend to $\mathsf{C}$?'.

## 3.1 Density and extension by colimits

In this chapter we will show how $\mathsf{C}$ sits inside $\mathcal{P}\mathrm{sh}(\mathsf{C})$.

**3.5.** Given a category $\mathcal{E}$ and a subcategory $i : \mathsf{C} \hookrightarrow \mathcal{E}$ we say that the $\mathsf{C}$ together with $i$ are a **dense subcategory** in $\mathcal{E}$ if every object $e \in \mathcal{E}$ is some colimit of a diagram valued in $\mathsf{C}$. In other words, there is a diagram $d : \mathsf{D}_e \to \mathsf{C}$ such that $\mathsf{colim}(\mathsf{D}_e \xrightarrow{d} \mathsf{C} \hookrightarrow \mathcal{E}) = e$ for each object $e \in \mathcal{E}$. Similarly we can talk about an arbitrary functor $F : \mathsf{C} \to \mathcal{E}$ being **dense**.

**dense subcategory**

**dense functor**

**3.6.** For every object $e \in \mathcal{E}$ there is a slice category $\mathcal{E}_{/e}$ with the canonical projection $\mathcal{E}_{/e} \to \mathcal{E}$ sending $f \mapsto \mathsf{dom}(e)$. The colimit over this diagram is $e$, so $\mathsf{colim}(\mathcal{E}_{/e} \xrightarrow{\mathsf{dom}} \mathcal{E}) = e$. The proof is very similar to the Yoneda lemma and relies on the fact that any $f \in \mathcal{E}_{/e}$ factors as $f \circ \mathsf{id}_e$. Recall that a cocone over $\mathcal{E}_{/e} \to \mathcal{E}$ into $e'$ is a natural transformation $\theta : \mathsf{dom} \to \Delta e'$ sending $f \in \mathcal{E}_{/e}$ to $\theta_f : c \to e'$. Then $\theta_{\mathsf{id}_e} : e \to e'$, so to show that $e$ is the colimit we just have to check that this map is unique. Using $f = f \circ \mathsf{id}_e$ and naturality gives

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, "\theta_f"] & e \arrow[d, "\theta_{\mathsf{id}_e}"] \\
e' \arrow[r, equal] & e'
\end{tikzcd}$$

so $\theta_f = \theta_{\mathsf{id}_e} \circ f$. Each natural transformation gives us a map $\theta_{\mathsf{id}_e} : e \to e'$. Furthermore each such map produces a natural transformation, hence $e = \mathsf{colim}(\mathcal{E}_{/e} \to \mathcal{E})$. And so we have that every category in canonically dense in itself.

**3.7.** We now wish to restrict the canonical diagram $\mathcal{E}_{/e} \to \mathcal{E}$ to a subcategory $\mathsf{C}$, this will give us a canonical way to obtain $e$ as a colimit valued in $\mathsf{C}$ as in 3.5. Instead of doing this in an ad-hoc fashion, i.e. restricting the projection functor $\mathcal{E}_{/e} \to \mathcal{E}$ by considering the subcategory of $\mathcal{E}_{/e}$ induced by the subcategory $\mathsf{C} \hookrightarrow \mathcal{E}$, we generalize the slice construction.

Given two subcategories $\mathsf{S} \hookrightarrow \mathcal{E} \hookleftarrow \mathsf{T}$, their **comma category** is the category $\mathcal{E}_{\mathsf{S}/\mathsf{T}}$ whose

**comma category**

- objects are morphisms $f : s \to t$ in $\mathcal{E}$ where $s \in \mathsf{S}$, $t \in \mathsf{T}$
- morphisms $(u, v) : f \to f'$ are pairs $u \in \mathsf{S}$ and $v \in \mathsf{T}$ such that the following square commutes

$$\begin{tikzcd}
s \arrow[r, "u"] \arrow[d, "f"] & s' \arrow[d, "f'"] \\
t \arrow[r, "v"] & t'
\end{tikzcd}$$

The comma category $\mathcal{E}_{\mathsf{S}/\mathsf{T}}$ comes with canonical maps $\mathsf{S} \xleftarrow{p} \mathcal{E}_{\mathsf{S}/\mathsf{T}} \xrightarrow{q} \mathsf{T}$ sending arrows $f$ to their domain and codomain respectively. The ordinary slice $\mathcal{E}_{/e}$ is obtained by letting $\mathsf{S} = \mathcal{E}$ and $\mathsf{T} = \{e\}$ the discrete subcategory of $\mathcal{E}$ containing only $e$. Dually, the coslice $\mathcal{E}_{e/}$ is obtained from setting $\mathsf{S} = \{e\}$ and $\mathsf{T} = \mathcal{E}$.

Note: Often the construction of a comma category is generalized further to arbitrary functors $S : \mathsf{S} \to \mathcal{E}$ and $T : \mathsf{T} \to \mathcal{E}$.

The assignment $x \mapsto \mathcal{E}_{\mathsf{C}/x}$ from $\mathcal{E}$ to $\mathcal{C}\mathrm{at}$ is functorial in $x$. Indeed from $f : x \to y$ in $\mathcal{E}$ we get a functor $\mathcal{E}_{\mathsf{C}/x} \to \mathcal{E}_{\mathsf{C}/y}$ sending $g : c \to x$ in the slice over $x$ to $gf : c \to y$ in the slice over $y$ as displayed below:

$$\begin{tikzcd}
c \arrow[dr, "g"] \arrow[dd, "h"'] \arrow[drr, dashed, bend left, "fg"] & & \\
 & x \arrow[r, "f"] & y \\
c' \arrow[ur, "g'"'] \arrow[urr, dashed, bend right, "fg'"'] & &
\end{tikzcd}$$

The desired restriction of $\mathcal{E}_{/e}$ to a subcategory $\mathsf{C}$ is now $\mathcal{E}_{\mathsf{C}/e}$. For an subcategory $\mathsf{C}$ of a category $\mathcal{E}$ we functorially obtain the diagram $\mathsf{dom} : \mathcal{E}_{\mathsf{C}/e} \to \mathsf{C}$. An object $e \in \mathcal{E}$ is said to be in the closure of a subcategoy $\mathsf{C} \hookrightarrow \mathcal{E}$ if $e = \mathsf{colim}(\mathcal{E}_{\mathsf{C}/e} \to \mathsf{C} \to \mathcal{E})$. A subcategoy $\mathsf{C}$ is said to be dense in $\mathcal{E}$ if every object $e$ is in the closure of $\mathsf{C}$. 3.7 shows that every category is dense in itself.

Using the Yoneda embedding we can consider $\mathsf{C} \xhookrightarrow{\mathbf{y}} \mathcal{P}\mathsf{sh}(\mathsf{C})$ as a subcategory. With this we can associate to a preasheaf $x \in \mathcal{P}\mathsf{sh}(\mathsf{C})$ the **category of elements** $\mathcal{P}\mathsf{sh}(\mathsf{C})_{\mathsf{C}/x}$. Explicitly: the objects of $\mathcal{P}\mathsf{sh}(\mathsf{C})_{\mathsf{C}/x}$ are morphisms with representable domain $f : \mathbf{y}c \to x$ and a morphism between $f : \mathbf{y}c \to x$ and $f' : \mathbf{y}c' \to x$ consist of an $h : c \to c'$ such that $f'\mathbf{y}(h) = f$.

**category of elements**

**3.8.** *Every presheaf $x \in \mathcal{P}sh(\mathsf{C})$ is the colimit of the diagram $\mathcal{P}sh(\mathsf{C})_{\mathsf{C}/x} \xrightarrow{p} \mathsf{C} \xrightarrow{\mathbf{y}} \mathcal{P}sh(\mathsf{C})$.*

In what follows we will often take colimits of diagrams like $\mathcal{P}\mathsf{sh}(\mathsf{C})_{\mathsf{C}/x} \xrightarrow{p} \mathsf{C} \xrightarrow{F} \cdot$ with $F$ some functor. In this case we will write $\mathsf{colim}(Fp)$ as $\mathsf{colim}_{f:\mathbf{y}c\to x}(F(c))$, understanding $f$ to vary over the category $\mathcal{P}\mathsf{sh}(\mathsf{C})_{\mathsf{C}/x}$. The diagram above is of this type and with this convention we write $\mathsf{colim}_{f:\mathbf{y}c\to x}\, \mathbf{y}c$ for the colimit.

*Proof.* Note that the functor $\mathcal{P}\mathsf{sh}(\mathsf{C})_{\mathsf{C}/x} \xrightarrow{\mathbf{y}p} \mathcal{P}\mathsf{sh}(\mathsf{C})$ sends sends arrows $f : \mathbf{y}c \to x$ to their domain $\mathbf{y}c$. The comma category $\mathcal{P}\mathsf{sh}(\mathsf{C})_{\mathsf{C}/x}$ also comes equipped with another functor $q : \mathcal{P}\mathsf{sh}(\mathsf{C})_{\mathsf{C}/x} \to \{x\} \subset \mathcal{P}\mathsf{sh}(\mathsf{C})$ sending $f$ to its codomain, this functor is isomorphic to $\Delta(x) : \mathcal{P}\mathsf{sh}(\mathsf{C})_{\mathsf{C}/x} \to \mathcal{P}\mathsf{sh}(\mathsf{C})$. The arrows $f$ themselves form a natural transformation between these functors, yielding a cocone $\lambda : \mathbf{y}p \Rightarrow \Delta(x)$ with legs $\lambda_f = f : \mathbf{y} \circ p \to x$.

This cocone is $\lambda$ is universal, indeed suppose we have another cocone $\mu : \mathbf{y}p \Rightarrow \Delta(y)$ consisting of maps $\mu_f : \mathbf{y}c \to y$ one for each $f : \mathbf{y}c \to x$. We produce a morphism of presheaves $F_\mu : x \to y$ levelwise: for each $c \in \mathsf{C}$ we define $F_\mu(c) : x(c) \to y(c)$. Consider $f \in x(c)$, by Yoneda this corresponds to $f : \mathbf{y}c \to x$, an object of $\mathcal{P}\mathrm{sh}(\mathsf{C})_{\mathsf{C}/x}$. Then the leg $\mu_f : \mathbf{y}c \to y$ associated with $f$ in the cocone corresponds, again by Yoneda, to an element $\mu_f \in Y(c)$ which will be the image $s$ under $F_\mu(c)$. This defines the data of a morphism of presheaves and the naturality of the cocone ensures that $F_\mu$ is a natural transformation.

Conversely, given a map of presheaves $G : x \to y$ we can extend the standard cocone $\lambda$ to a cone $\overline{G}$ over $y$ by setting $\overline{G}_f := G \circ \lambda_f = G \circ f$.

These operations are inverse to each other, indeed suppose we start with $G : x \to y$ a morphism of presheaves, then construct $\overline{G} : \mathcal{P}\mathrm{sh}(\mathsf{C})_{\mathsf{C}/x} \Rightarrow \Delta y$, and from this produce $F_{\overline{G}} : x \to y$. Then $G$ and $F_{\overline{G}}$ are equal when $G(c)$ and $F_{\overline{G}}(c)$ are equal maps $x(c) \to y(c)$ for each $c \in \mathsf{C}$. An $f \in x(c)$ corresponds to $f : \mathbf{y}c \to x$ and so $G(c)(f) = G \circ f$ on the other hand $F_{\overline{G}}(c)(f) = \overline{G}_f = G \circ f$.

Similarly we show that any cone $\mu : \mathbf{y}p \Rightarrow \Delta y$ is equivalent to $\overline{F_\mu}$. Indeed $\overline{F_\mu}_f = F_\mu \circ f$

This amounts to showing that $\mu_f = \overline{F_\mu}_f$, but $\overline{F_\mu}_f = F_\mu \circ f = F_\mu(c)(f) = \mu_f$.

□

We can now completely characterize the relation between $\mathsf{C}$ and $\mathcal{P}\mathrm{sh}(\mathsf{C})$. Indeed the category $\mathsf{C}$ is dense in $\mathcal{P}\mathrm{sh}(\mathsf{C})$ along the Yoneda embedding. Conversely, every diagram in $\mathsf{C}$ has a colimit in $\mathcal{P}\mathrm{sh}(\mathsf{C})$ when extended along $\mathbf{y}$. The following theorem shows that $\mathbf{y} : \mathsf{C} \to \mathcal{P}\mathrm{sh}(\mathsf{C})$ is the universal extension of $\mathsf{C}$ with these properties.

**3.9.** Given a small category $\mathsf{C}$ and a locally small $\mathcal{E}$. For any functor $F : \mathsf{C} \to \mathcal{E}$ we can define the evaluation at $F$ functor

$$F^* : \mathcal{E} \to \mathcal{P}\mathrm{sh}(\mathsf{C}), \quad F^*(e) = \big(c \mapsto \mathsf{Hom}_{\mathcal{E}}(F(c), e)\big)\,.$$

If $\mathcal{E}$ admits small colimits then $F^* : \mathcal{E} \to \mathcal{P}\mathrm{sh}(\mathsf{C})$ has a left adjoint

$$F_! : \mathcal{P}\mathrm{sh}(\mathsf{C}) \to \mathcal{E}, \quad F_!(x) = \mathsf{colim}(\mathcal{P}\mathrm{sh}(\mathsf{C})_{\mathsf{C}/x} \to \mathsf{C} \xrightarrow{F} \mathcal{E}) = \underset{f:\mathbf{y}c \to x}{\mathsf{colim}}\, F(c)$$

such that the diagram on the left commutes

$$\begin{tikzcd} & \mathcal{P}\mathrm{sh}(\mathsf{C}) \arrow[d, "F_!"] \\ \mathsf{C} \arrow[ur, hook, "\mathbf{y}"] \arrow[r, "F"'] & \mathcal{E} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}\mathrm{sh}(\mathsf{C}) \arrow[d, bend right, "F_!"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{E} \arrow[u, bend right, "F^*"'] \end{tikzcd}$$

This shows that $\mathbf{y} : \mathsf{C} \hookrightarrow \mathcal{P}\mathrm{sh}(\mathsf{C})$ is the universal map from $\mathsf{C}$ into a cocomplete category: Every other map $F : \mathsf{C} \to \mathcal{E}$ induces a unique $F_! : \mathcal{P}\mathrm{sh}(\mathsf{C}) \to \mathcal{E}$. We call the induced map the $F_!$ the **extension of $F$ by colimits**.

**extension by colimits**

*Proof.* Abusing notation we can write the $\mathsf{colim}(F \circ \mathsf{dom})$ as $\mathsf{colim}_{\mathbf{y}c \to x} F(\mathbf{y}c)$. Then for arbitrary $x \in \mathcal{P}\mathrm{sh}(\mathsf{C})$ and $e \in \mathsf{E}$

$$\begin{aligned}
\mathsf{Hom}(F_!(x), e) &= \mathsf{Hom}_{\mathcal{E}}(\underset{f:\mathbf{y}c \to x}{\mathsf{colim}} F(c), e) && \text{def of } F_! \\
&= \lim_{f:\mathbf{y}c \to x} \mathsf{Hom}_{\mathcal{E}}(F(c), e) && \mathsf{Hom} \text{ preserves limits} \\
&= \lim_{f:\mathbf{y}c \to x} F^*(e)(c) && \text{def of } F^* \\
&= \lim_{f:\mathbf{y}c \to x} \mathsf{Hom}_{\mathcal{P}\mathrm{sh}(\mathsf{C})}(\mathbf{y}c, F^*(e)) && \text{Yoneda lemma} \\
&= \mathsf{Hom}_{\mathcal{P}\mathrm{sh}(\mathsf{C})}(\mathsf{colim}(\mathcal{P}\mathrm{sh}(\mathsf{C})_{\mathsf{C}/x} \to \mathsf{C} \xrightarrow{\mathbf{y}}), F^*(e)) && \mathsf{Hom} \text{ preserves limits} \\
&= \mathsf{Hom}_{\mathcal{P}\mathrm{sh}(\mathsf{C})}(x, F^*(e)) && \text{By 3.8.}
\end{aligned}$$

On a representable $\mathbf{y}c$ we have $F_!(\mathbf{y}c) = \mathsf{colim}(\mathcal{P}\mathrm{sh}(\mathsf{C})_{\mathsf{C}/\mathbf{y}c} \to \mathsf{C} \xrightarrow{F} \mathcal{E})$ but $\mathsf{id}_{\mathsf{C}}$ is final in $\mathcal{P}\mathrm{sh}(\mathsf{C})_{\mathsf{C}/\mathbf{y}c}$ and so $F_!(\mathbf{y}c) = \mathsf{colim}(1 \xrightarrow{c} \mathsf{C} \xrightarrow{F} \mathcal{E}) \cong F(c)$. □

## 3.2 Presentable categories

Inside the category of sets we find the subcategory of finite sets $\mathcal{S}\mathrm{et}_{\mathsf{fin}}$. When working with other categories such as topological spaces, groups of vector spaces we would like to find an analogous notion of 'finite object'. These should not be literally finite, for example the group $\mathbb{Z}$ and the vector space $\mathbb{R}^2$ should be finite in some other sense. The category of groups and the category of vector spaces admit a forgetful-free adjunction to $\mathcal{S}\mathrm{et}$ and the previously mentioned objects are both free over a finite set. We could try to define 'finite objects' along these lines but this suffers from some deficiencies. Instead, taking a cue from the definition of a compact space, the appropriate notion will be defined in terms of the objects in the category in question. In a sense, we get a notion of 'finite with respect to the objects of the category' or 'consists of a finite amount of other objects of the category'. This condition is known by a variety of names such as presentable or compact.

**3.10.** It will be helpful to recall the definition of a compact subspace and see how it translates to the categorical setting. We can associate to a space $X$ a poset (i.e. category) of opens $\mathcal{O}(X)$ and inclusions between opens. In categorical terms an open $K \in \mathcal{O}(X)$ is compact precisely if we can associate to any diagram $U_\bullet : I \to \mathcal{O}(X)$ such that $K \hookrightarrow \mathsf{colim}_{i \in I} U_i$ a finite subcategory $J \hookrightarrow I$ such that $K \hookrightarrow \mathsf{colim}_{i \in J} U_i$. One can easily verify that this is equivalent to the standard definition. We can replace the diagram $U_\bullet$ by another diagram $V_\bullet : \mathcal{P}_{\mathsf{fin}}(I) \to \mathcal{O}(X)$ such that $V_J = \mathsf{colim}_{i \in J} U_i$ with the same colimit. Then the compactness condition simplifies to: if $K \hookrightarrow \mathsf{colim}_J V_J$ then there is a $J \subset_{\mathsf{fin}} I$ such that $K \hookrightarrow V_J$.

**3.11.** The passage from $U_\bullet$ to $V_\bullet$ replaces an normal colimit with a so called directed colimit which has close relation to filtered colimits and sequential colimits which we define in a moment. To do this we first recall that for any cardinal $\kappa$

- A category is $\kappa$-**small** if it has less then $\kappa$ arrows, i.e. $|\mathcal{M}\mathrm{or} D| < \kappa$.

**small, filtered, directed and chain categories**

- A category $\mathsf{D}$ is $\kappa$-**filtered** if any $\kappa$-small subcategory $\mathsf{D}'$ has a cocone.
- A category $\mathsf{D}$ is $\kappa$-**directed** if any $\kappa$-small subcategory $\mathsf{D}' \hookrightarrow \mathsf{D}$ has a colimiting cocone.
- A category is a $\kappa$-**chain** if the category is equivalent to the ordinal $\kappa$.

In the case of $\kappa = \omega$ we speak just of filtered and directed categories. An category colimit over a diagram with shape $\mathsf{D}$ is called; $\kappa$-small if $\mathsf{D}$ is $\kappa$-small; $\kappa$-filtered if $\mathsf{D}$ is $\kappa$-filtered; $\kappa$-directed if $\mathsf{D}$ is $\kappa$-directed; and $\kappa$-sequential if $\mathsf{D}$ is a $\kappa$-chain.

**3.12.** We will consider colimits for which the shapes are given by the class of categories defined above, in this case we will call the colimit after the shape of its diagram. For example a finitely filtered colimit in $\mathsf{C}$ will be a colimit of a diagram $p : \mathsf{D} \to \mathsf{C}$ where $\mathsf{D}$ is finitely filtered.

- A category has $\kappa$-filtered colimits iff it has $\kappa$-directed colimits.
- A functor $F$ between such categories preserves $\kappa$-filterd colimits iff it preserves $\kappa$-directed colimits.
- A category has (finite) filtered colimits iff it has chain colimits, a functor $F$ between such categories preserves filtered colimits iff it preserves chain colimits.

**3.13.** An object $A$ in a category $\mathsf{C}$ is said to be $\kappa$-**presentable** if the functor $\mathsf{Hom}(A, -) : \mathsf{C} \to \mathcal{S}\mathrm{et}$ preserves $\kappa$-filtered colimits. When speaking of presentable objects we sometimes leave $\kappa$ implicit.

**presentable object**

**3.14.** This somewhat mysterious condition says that if we can produce an $\kappa$-filtered colimit $\mathsf{colim}_i A_i$ then $\mathsf{Hom}(A, \mathsf{colim}_i A_i) = \mathsf{colim}_i \mathsf{Hom}(A, A_i)$. In other words if there is a map $f : A \to \mathsf{colim}_i A_i$ then $f$ factors through an leg of the colimit.

**3.15.** The most familiar setting in which we encounter presentable objects is in the setting of topology. Let $\mathcal{O}(X)$ be the poset of opens of a topological space $X$, then an open $C \in \mathcal{O}(X)$ is compact precisely when it is presentable in $\mathcal{O}(X)$. It is for this reason that a presentable object is also called a **compact object**.

Consider any cover $\{U_i\}_{i \in I}$ of $C$ and write $U = \bigcup_i U_i$. That $U$ covers $C$ means that there is an inclusion $C \hookrightarrow U$. The opens form a diagram $U_\bullet : I \to \mathcal{O}(X)$ which we can replace by the directed diagram $V_\bullet : \mathcal{P}_{\mathsf{fin}}(I) \to \mathcal{O}(X)$ such that $V_S = \bigcup_{i \in S} U_i$, this clearly has the same colimit $\bigcup_S V_S = U$. Assuming that $C$ is presentable yields an $S$ such that $C \hookrightarrow \bigcup_{i \in S} U_i$, this is precisely a finite cover over $C$ showing that $C$ is compact. Conversely if $C$ is compact any directed covering $V_\bullet : J \to \mathcal{O}(X)$ produces is in particular a cover yielding a finite directed subcover $V_\bullet : J' \to \mathcal{O}(X)$ where $J'$ is a finite subcategory of $J$. By assumption on $J$ we find a colimiting cone $j \in J$ of $J'$ and then $V_j$ covers $C$.

**3.16.** Intuitively we might say that we can break up finitely presentable objects $C$ into a finite amount of pieces by supplying a filtered diagram $U_\bullet$ and a 'covering' $C \to \mathsf{colim}_i U_i$. For example a set $K$ is finitely presentable in the category of sets if it is finite.

**3.17.** An cocomplete category $\mathsf{C}$ is an $\kappa$-**presentable** if

**presentable category**

(i) It has a small set of $\mathsf{A}$ of $\kappa$-presentable objects.

(ii) Every object is an $\kappa$-directed colimit of objects in $\mathsf{A}$.

An category is presentable if it is $\kappa$-presentable for some $\kappa$.

**3.18.** If $\mathsf{C}$ is an category with a full subcategory $\mathsf{A}$, we let $\mathbf{y}|_{\mathsf{A}}$ denote the restriction of the Yoneda functor to morphisms in $\mathsf{A}$. To be precise

$$\mathbf{y}|_{\mathsf{A}}(X) : A \mapsto \mathsf{Hom}(A, X), \qquad \text{where } A \in \mathsf{A}$$

**3.19.** *Let* $\mathsf{A}$ *be a small, full subcategory of* $\mathsf{C}$ *then* $\mathbf{y}|_{\mathsf{A}}$

*(i) if full and faithful iff* $\mathsf{A}$ *is dense*

*(ii) preserves* $\kappa$*-directed colimits iff every object in* $\mathsf{A}$ *is presentable*

*Proof.* [Adá+94, 1.26] □

**3.20.** *If* $\mathsf{C}$ *is* $\kappa$*-presentable and* $\mathsf{A}$ *is the subcateogy of* $\kappa$*-presentable objects then the above shows that* $\mathsf{C}$ *is a reflexive subcategory of* $\mathcal{P}_{sh}(\mathsf{A})$.

We will use the following important properties about presentable categories

(i) By definition every presentable category has small colimits.

(ii) Since it is the reflexive localization of an presheaf category it also has all small limits.

(iii) Any cocontinuous functor between presentable categories has an right adjoint.

*Proof.* For (iii) see [Rie17, 4.6.17] and [Adá+94, 1.58]. □

## 3.3 Topoi

Topoi arose historically as the collection of étale maps of topological spaces. An étale map over $X$ is a contiunuous map $p : E \to X$ which is a local homeomorphism. This means that any point $x \in E$ admits an open $U \in \mathcal{O}(X)$ such that $x \in U$ and $p|_V$ is an homeomorphism onto its image. Equivalently such an etalé map $f$ corresponds to a functor from $\mathcal{O}(X)$ to $\mathcal{S}\mathrm{et}$ assigning to each open $U$ the set of cross sections of $p$. Categorically these functors are characterized by a gluing condition of open covers.

**3.21.** Given a topological space $X$ write $\mathcal{O}(X)$ for the **poset of opens** of $X$ ordered by inclusion. Recall that a presheaf on $\mathcal{O}(X)$ is a functor $\mathcal{O}(X)^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$, in this case we will also speak of **a presheaf on** $X$ understanding that we really mean the poset of opens. For an presheaf $F$ on $X$ and $x \in F(U)$ we will write $x|_V$ for $F(i)(x)$ where $i : V \hookrightarrow X$ is the canonical inclusion (if it exists).

**poset of opens**

**3.22.** Let $F : \mathcal{O}(X)^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$ be an presheaf on $X$ and let $\{U_i\}_{i \in I}$ be a cover of $X$. Then any $x \in F(X)$ induces an family $(x_i)|_{i \in I}$ with $x_i = x|_{U_i} \in F(U_i)$ such that $x_i|_{U_j} = x_j|_{U_i}$ for all $i, j \in I$, such a family is called a **compatible family**.

**compatible family**

**3.23.** It is very useful to consider such a compatible family as a decomposition of $x$. A common idea in topology and geometry is to decompose an global object into smaller local objects and attempt to study objects that way. The sheaf condition below says that we can go the other way, any collection of local data (i.e. compatible family) determines a unique global datum. It is for this reason that topoi can be thought of as a bridge between local and global reasoning.

**3.24.** A **sheaf** on a topological space $X$ is a presheaf $F$ such that for any open cover $\{U_i\}_{i \in I}$ and compatible family $(x_i)_{i \in I}$ there is a unique **amalgamation** $x \in F(X)$ such that $x_i = x|_{U_i}$. We can separate this condition by saying that

**sheaf**

(i) $F$ is separated if each compatible family has at most one amalgamation

(ii) $F$ is fibrant if each compatible family has at least one amalgamation

(iii) $F$ is a sheaf if it is both separated and fibrant

**3.25.** The **topos of sheaves** on a topological space $X$ written $\mathcal{S}\mathrm{h}(X)$ is the collection of sheaves on $X$

**topos of sheaves**

**3.26.** Any continuous map $f : X \to Y$ induces a pair of adjoint functors

$$\begin{tikzcd} \mathcal{S}\mathrm{h}(X) \arrow[r, shift left, "f_*"] & \mathcal{S}\mathrm{h}(Y) \arrow[l, shift left, "f^*"] \end{tikzcd} \qquad \begin{tikzcd} f^*E \arrow[r] \arrow[d, red] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

here the **direct image** functor $f_*$ is defined by $f_*F : V \mapsto F(f^{-1}(V))$. The **inverse image** functor is better understand if we represent a sheaf $G$ on $Y$ by an étale map $p_G : E \to Y$, then the $f^*G$ is represented by the red pullback as displayed above right.

**3.27.** It was then realized that this situation generalizes to arbitrary categories not just categories of opens. Then the gluing conditions of open covers becomes can be encoded by a Grothendieck topology.

**3.28.** A **sieve** $S$ on $c \in \mathsf{C}$ is an downward closed collection of morphisms into $c$. So

**sieve**

(i) $f_i \in S$ then $f_i : c_i \to c$.

(ii) if $f_i : c_i \to c \in S$ and $g : d \to c_i$ then $gf_i \in S$.

Equivalently a sieve is a subfunctor of $\mathbf{y}c$, i.e. a mono $S \to \mathbf{y}c$.

**3.29.** For any collection of morphisms $\mathcal{S}$ into $c$ we can consider $S$ **the sieve generated by** $\mathcal{S}$. This is just the closure of $\mathcal{S}$ under precomposition or equivalently the smallest sieve containing $\mathcal{S}$.

**3.30.** Let $S$ be a sieve on $c$ and let $f : d \to c$ be a morphism then the pullback sieve $f^*S$ on $d$ are those maps in $S$ factoring through $f$. This is just the pullback of $S \to \mathbf{y}c$ along $\mathbf{y}f : \mathbf{y}d \to yc$ in $\mathcal{P}\mathrm{sh}(\mathsf{C})$ which exists because it has finite limits.

**3.31.** An **Grothendieck topology** $J$ assigns to every object $c \in \mathsf{C}$ a collection of sieves $J(c)$ which are said to be **covering sieves**. Subject to the following requirements

**Grothendieck topology**

**maximal sieve** the maximal sieve of all maps into $c$, corresponding to the identity $\mathbf{y}c \to \mathbf{y}c$, covers $c$.

**pullback stability** if $S$ covers $c$ and $f : c' \to c$ is a morphism then $f^*S$ covers $c'$.

**intersection** two sieves $S$ and $T$ both cover $c$ if and only if $S \cap T$ covers $c$, here $S \cap T$ is the pullback in $\mathcal{P}\mathrm{sh}(\mathsf{C})$.

**transitivity** if $S$ is a sieve on $c$ and let $\mathcal{T} := \{f_i : c_i \to c \mid f_i^* S \text{ covers } c_i\}$ then if the sieve generated by $\mathcal{T}$ covers $c$ then also $S$ covers $c$.

**3.32.** A **site** $(\mathsf{C}, J)$ is category $\mathsf{C}$ together with a Grothendieck topology $J$. **site**

**3.33.** Let $J$ be a Grothendieck topology, then such a Grothendieck topology corresponds to a collection of monomorphism $\mathcal{J}$ in $\mathcal{P}\mathrm{sh}(\mathsf{C})$, i.e. every sieve on $c$ determines a monomorphism into $\mathbf{y}c$. Then a presheaf $F$ in $\mathcal{P}\mathrm{sh}(\mathsf{C})$ is an sheaf on $(\mathsf{C}, J)$ if it is an $\mathcal{J}$-sheaf in the sense of 2.36.

**3.34.** The category $\mathcal{S}\mathrm{h}(\mathsf{C}, J)$ of sheaves on the site is equivalent to the full category of $\mathcal{J}$-sheaves. The inclusion $\mathcal{S}\mathrm{h}(\mathsf{C}, J) \hookrightarrow \mathcal{P}\mathrm{sh}(\mathsf{C})$ has a left adjoint called the **associated sheaf functor a** : $\mathcal{P}\mathrm{sh}(\mathsf{C}) \to \mathcal{S}\mathrm{h}(\mathsf{C}, J)$. As a left adjoint **a** preserves all colimits, but it also preserves all finite limits.

*Proof.* For the definition of **a** and the fact that it preserves finite limits see [MM12, V.3]. □

**3.35.** A **topos** is a category equivalent to the category of sheaves on a site. This means that any topos is a left exact localization of a preasheaf category. **topos**

A **morphism of topoi** $f : \mathcal{E} \to \mathcal{D}$ or an **geometric morphism** is an adjoint pair

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left=2, "f_*"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r] & \mathcal{D} \arrow[l, shift left=2, "f^*"]
\end{tikzcd}$$

such that $f^*$ additionally preserves finite limits. Thus we get a category of topoi.

## 3.4 Logoi

Spaces are usually described by their point set topology, i.e. a set of points and a collection of opens. The collection of opens form what is called a frame: a poset under inclusion with arbitrary joins (unions) and finite meets (intersections) satisfying a distributivity law. What is essential to note is that the spatial notion of continuity is then encoded contravariantly by the lattice of opens. What we mean by this is that a map $f : X \to Y$ is continuous precisely when $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a well defined morphism of frames. This is part of a rich duality between geometry and algebra which has many more extensions.

The theory of topoi are intended to be generalized spaces. This suggests that they might be dually understood in an algebraic way. Following Joyal and Anel [AJ19] we call the dual of the category of topoi the category of **logoi** and we will define them now.

**3.36.** The limits of a category (when they exists) are always in some canonical relation with another. For example the terminal object **1** is the unit of the binary product $\times$. Limits can be compared this way because their universal properties have the same 'handedness'.

In contrasts limits and colimits do not have to interact in any meaningful way. For example the following 'high school distributivity laws' a la Tarski do not hold in a general category with the appropriate limits and colimits.

- $X \times (A + B) \cong X \times A + X \times B$
- $X \times 0 \cong 0$

In the category $\mathcal{S}\mathrm{et}$ of sets these laws do hold. Logoi can be characterized as categories for which some distributively condition of limits over colimits hold.

**3.37.** The link with arithmetic in $\mathbb{N}$ can actually be made slightly more precise. Let $\mathcal{S}\mathrm{et}_{\mathsf{fin}}$ be the full subcategory of $\mathcal{S}\mathrm{et}$ spanned by the finite sets. This category inherits finite limits and colimits from $\mathcal{S}\mathrm{et}$. There is an cardinality map $|-| : \mathcal{S}\mathrm{et}_{\mathsf{fin}} \to \mathbb{N}$ sending a set to its cardinality. The structure $(+, \times, 0, 1, \mathbb{N})$ is an symmetric rig which is a symmetric ring without negatives[nLa19]. Correspondingly the category $\mathcal{S}\mathrm{et}_{\mathsf{fin}}$ can be given an approriate categorical notion of a symmetric rig $(+, \times, \mathbf{0}, \mathbf{1}, \mathcal{S}\mathrm{et}_{\mathsf{fin}})$. Then the cardinality map is a symmetric rig morphism.

**3.38.** A category with all finite limits is also called left exact or **lex category**, this terminology is justified because functors between abelian categories are left exact precisely when they preserve finit limits. Similarly a category preserving small colimits is called cocomplete or a **cc category** for short. In this chapter we will often consider categories with finite limits and all small colimits which can be abbreviated as **lex cc categories**.

**lex category**

**cc category**

**lex cc categories**

**3.39.** It would be too naive to assert the laws like 3.36 as is. For one these laws do not relate pullbacks and pushouts. Instead the proper generalization of the distributivity laws above start with the following property.

**3.40.** In the following we will often consider an lex cc category $\mathcal{E}$ which will often remain implicit. We will also refer to diagrams $X_\bullet$ by which we mean a functor $\mathsf{I} \to \mathcal{E}$ sending $i \mapsto X_i$. The colimit will be often written as just $X = \mathsf{colim}_i X_i$. When we talk about multiple such diagrams the category $\mathsf{I}$ is usually the same.

**3.41.** A lex cc category is said to have **universal colimits** if any diagram $X_\bullet$ over $B$ satisfies

**universal colimits**

$$A \times_B (\operatorname*{colim}_i X_i) \cong \operatorname*{colim}_i (A \times_B X_i)$$

**3.42.** This definition has the following reformulation. Let $f : A \to B$ be some morphism in a category with pullbacks, then the composition with $f$ functor $(f \circ -) : \mathsf{C}_{/A} \to \mathsf{C}_{/B}$ has an right adjoint, called the **change of base** functor, given by pullback

**change of base**

$$f^* : \mathsf{C}_{/B} \to \mathsf{C}_{/A}, \qquad f^*(X) = A \times_B X$$

A category has universal colimits if all base change functors preserve colimits. In the example pictured below where $X = \mathsf{colim}_i X_i$ this means that $f^* X = \mathsf{colim}_i f^* X_i$.

$$\begin{tikzcd}
& f^*X_0 \arrow[dl] \arrow[dr] & & & X_0 \arrow[dl] \arrow[dr] & \\
f^*X_1 \arrow[dr] & & f^*X_2 \arrow[dl] & X_1 \arrow[dr] & & X_2 \arrow[dl] \\
& f^*X \arrow[d] & & & X \arrow[d] & \\
& A \arrow[rrr, "f"] & & & B &
\end{tikzcd}$$

**3.43.** A special case of this definition is when $B = X$ such that the pullback is along a map $f : Y \to X$ into the colimit. Pulling back $f$ along all legs of the colimit $X = \mathsf{colim}_i X_i = X$ produces a diagram like the one below where $Y_i = f^*X_i$.

$$\begin{tikzcd}
Y_0 \arrow[r] \arrow[d] \arrow[drr, bend right] \arrow[dr, phantom, "\ulcorner", very near start] & Y_1 \arrow[d] \arrow[dr] & \\
X_0 \arrow[r] \arrow[drr, bend right] & X_1 \arrow[dr] & Y \arrow[d, "f"] \\
& & X
\end{tikzcd}$$

All squares with side $f$ are pullback squares. By the pullback pasting law the far end square is also a pullback square, as indicated. That colimits are universal means that the colimit of the induced diagram $Y_\bullet$ is $Y$. The natural transformation $Y_\bullet \Rightarrow X_\bullet$ given by the vertical arrows in this diagram has the property that all naturality squares are pullback squares. A natural transformation with this property is a **cartesian natural transformation**. The property of descent which we will introduce now is a kind of reciprocal to this statement.

**cartesian natural transformation**

**3.44.** The **category of descent data** $\mathsf{Desc}(X_\bullet)$ for diagram $X_\bullet : \mathsf{I} \to \mathcal{E}$ is the category of cartesian transformations $Y_\bullet \Rightarrow X_\bullet$ between diagrams $\mathsf{I} \to \mathcal{E}$. This category is the limit of the slice categories induced by the diagram

**category of descent data**

$$\mathsf{Desc}(X_\bullet) = \lim_i \mathsf{C}_{X_i}$$

To see this, notice that for any cone over $\mathsf{C}_{/X_\bullet}$ is a family of diagrams $F_i : \mathsf{Z} \to \mathsf{C}_{/X_i}$ such that any morphism $i \to j$ in the diagram relates diagrams by pullback $\mathsf{C}_{/X_j} \to \mathsf{C}_{/X_i}$. In particular for every object $z \in \mathsf{Z}$ do we get a an cartesian functor $i \mapsto F_i(z)$ determining a unique map $\mathsf{Z} \to \lim_i \mathsf{C}_{/X_i}$.

**3.45.** There is an adjoint relation between the descent data of a diagram $X_\bullet$ and an morphism over the colimit $Y \to X$.

$$\begin{tikzcd}
\mathcal{E}_{/X} \arrow[r, phantom, "\perp"] \arrow[rr, bend right, "\mathsf{dec}_{X_\bullet}"'] & & \mathsf{Desc}(X_\bullet) \arrow[ll, bend right, "\mathsf{glue}_{X_\bullet}"']
\end{tikzcd}$$

Here $\mathsf{dec}$ decomposes an object over the slice as described in 3.43. The gluing map $\mathsf{glue}$ takes the colimit of $Y_\bullet$ and uses it's universal property to obtain a unique map into $X$. This is an adjoint because if there is a map $\mathsf{colim}_i Y_i = Y \to Z$ over $X$ then this results in a maps $Y_\bullet \to Z_\bullet$ cartesian over $X_\bullet$.

**faithful descent**

**3.46.** The colimits of a diagram $X_\bullet$ are said to be of **faithful descent** if the decomposition functor is fully faithful. This means that an object in the slice $f : Y \to X$ can be decomposed as $Y_i = Y \times_X X_i$ and then be glued together again as the colimit of the diagram such that $\mathsf{colim}_i Y_i = Y$.

| faithful descent | effective descent |
|---|---|
| dec is ff | glue is ff |
| gluing after decomposition is identity | decomposition after gluing is identity |
| $\mathsf{colim}_i(Y \times_X X_i) = Y$ | $X_i \times_X (\mathsf{colim}_i Y_i) = Y_i$ |

**effective descent**

The colimits of a diagram $X_\bullet$ are said to be of **effective descent** if the gluing map is fully faithful. This means that if we glue an descent datum $Y_\bullet \Rightarrow X_\bullet$ together into $Y = \mathsf{colim}_i Y_i$ and then decompose it again we end up with the same descent datum.

**3.47.** *An category with pullbacks has universal colimits iff it has faithful descent for all diagrams.*

*Proof.* Faithful descent is just a special case of universal colimits, so we show the other direction. Suppose that we have faithful descent for all diagrams and a diagram $X_\bullet$ such that the colimit $X$ has a map $X \to B$ and a map $f : A \to B$. We can form the pullback $Y = A \times_B X$, then use faithfulness of the diagram $X_\bullet$ the compose $Y \to X$ to the components $Y_i \to X_i$. By composition of the pullback $Y_i$ is also the pullback of $X_i$ along $f$

$$\begin{tikzcd}
Y_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_i \arrow[d] \\
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

This means that $A \times_B (\mathsf{colim}_i X_i) = Y = \mathsf{colim}_i(Y \times_X X_i) = \mathsf{colim}_i(A \times_B X_i)$. So colimits are universal. $\square$

**full descent**

**3.48.** Colimits of diagram $X_\bullet$ have **full descent** if it's colimits are of faithful and effective descent. In this case the adjunction above becomes an equivalence of categories.

$$\mathcal{E}_{/\,\mathsf{colim}_i X_i} \cong \lim_i \mathcal{E}_{/X_i} = \mathsf{Desc}(X_\bullet)$$

The full descent condition for a diagram $X_\bullet$ with colimit $X$ can be concisely reformulated as follows. Consider an natural transformation of diagrams $\overline{\alpha} : Y_\bullet \to X_\bullet$ of shape $\mathsf{I}^{\rhd}$ obtained by adjoining a terminal object $\infty$ to $\mathsf{I}$. Let the restricted $\alpha = \overline{\alpha}|_I$ be an cartesian transformation and let $X_\infty = \mathsf{colim}_{i \in I} X_i$. Then the following are equivalent

(i) $Y_\infty$ is the colimit of $\mathsf{colim}_{i \in I} Y_i$

(ii) the whole $\overline{\alpha}$ is an cartesian transformation

*Proof.* (ii) → (i). Is faithfulness of descent. (i) → (ii). Is effectivity of descent. ☐

**3.49.** Intuitively the faithfulness of descent states that we can break up a large object, over $X$, into pieces, over $X_i$, and reason about it locally. Conversely, effectivity of descent is about our ability to assemble arbitrary objects from pieces, over $X_i$, into a whole, over $X$, without forgetting what happened on the pieces. Consider the following cartesian diagram in $\mathcal{S}\mathrm{et}$ where $\sigma$ swaps the elements.

$$\begin{tikzcd}
\{1,2\} \arrow[d] & \{1,2\} \sqcup \{1,2\} \arrow[l, "{[\mathsf{id},\sigma]}"'] \arrow[r, "{[\mathsf{id},\mathsf{id}]}"] \arrow[d] & \{1,2\} \arrow[d] \\
\{l\} & \{a,b\} \arrow[l] \arrow[r] & \{r\}
\end{tikzcd}$$

The pushout of the lower diagram and upper diagram (i.e. the lower and upper row respectively) are both one element sets yielding a map $\mathsf{id} : \mathbf{1} \to \mathbf{1}$. Clearly we can not retrieve the upper diagram by pulling back along this map. This shows that full descent fails in categories like $\mathcal{S}\mathrm{et}$. This example is due to Mathieu Anel[1].

**homotopically discrete colimit**

**3.50.** It turns out that the colimits in $\mathcal{S}\mathrm{et}$ that have effective descent are the ones that are **homotopically discrete**. A colimit $X_\bullet$ is homotopically discrete if the colimit in $\mathcal{S}\mathrm{et}$ agrees with the (homotopy) colimit in spaces. More precisely, write $\mathsf{Disc} : \mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$ for the discrete embedding of sets into spaces. Then a colimit $X_\bullet$ in $\mathcal{S}\mathrm{et}$ is homotopicaly discrete if $\mathsf{Disc}(\mathsf{colim}_i\, X_i) = \mathsf{colim}_i\, \mathsf{Disc}(X_i)$ where the last colimit is the homotopical colimit.

**3.51.** The above colimits of the diagram above fail to be homotopically discrete, to see this we can embed these sets discretely into $\mathcal{T}\mathrm{op}$ and then replace under homotopy equivalence. This yields the following cartesian diagram where all maps into $I$ are endpoint inclusions and $\sigma$ is the automorphism of the interval swapping the edges.

$$\begin{tikzcd}
I \sqcup I \arrow[d] & \{0,1\} \sqcup \{0,1\} \arrow[l, "{[i,\sigma \circ i]}"'] \arrow[r, "{[i,i]}"] \arrow[d] & I \sqcup I \arrow[d] \\
I & \{0,1\} \arrow[l] \arrow[r] & I
\end{tikzcd}$$

The pushout now yields $S^1$ and it's double cover map. Moreover in this diagram all maps can be recovered by (homotopy) pullback this double cover. This hints at the fact that in spaces all (homotopy) colimits are of full effective descent.

**3.52.** Topoi are inherently spacial object. By the duality between topology and algebra it has a dual representation as an algebraic object. Following Joyal & Anel [AJ19] we will call this dual presentation a logos.

**3.53.** A logos is an lex cc category $\mathcal{E}$ with universal colimits and effective descent for diagrams with homotopically discrete colimit.

**3.54.** The virtue of the above definition is that morphisms between logoi can be easily motivated. This is because descent is an internal property of a logos not requiring extra co-

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[1] `https://www.uwo.ca/math/faculty/kapulkin/seminars/hottestfiles/Anel-2019-05-2-HoTTEST.pdf`

herence conditions on functors. Therefore an **logos morphism** is an left exact cocontinous functor, in other words a functor preserving finite limits and small colimits.

**logos morphism**

**3.55.** Just as in a normal category the colimits are generated by coproducts and coequalizers the homotopically discrete colimits are generated by coproducts and coequalizers of congruences. A congruence on $X$ will be an internal equivalence relation i.e. an subject $R \to X \times X$ such that

(i) Reflexivity: the diagonal $\Delta : A \to A \times A$ factors through $R$

(ii) Transitivity: The pullback $R \times_A R$ factors through $R$ as subject of $A$.

(iii) Symmetry: the map $\sigma : R \to R$ is an isomorphism.

Intuitively such coequalizers are homotopically better behaved because 'identifying $a$ and $b$' means 'adding a path between $a$ and $b$' in $\mathcal{T}\mathrm{op}$. If those identification are then not transitive they add non trivial loops to the space. For more information see [AJ19].

**3.56.** *An lex cc category has descent for homotopically discrete colimits precisely when it has descent for coproducts and congruences.*

*Proof.* See Topo-logie ☐

**3.57.** The effective descent conditions of coproducts and congruences have a relation to the Giraud axioms as the following theorems show.

**3.58.** *An lex cc category $\mathcal{E}$ with universal colimits has*

*(i) effective descent for coproducts precisely when sums are disjoint.*

*(ii) effective descent for congruences precisely when equivalence relations are effective.*

*Proof.* Let $X_\bullet : I \to \mathcal{E}$ be a discrete diagram (i.e. $I$ is a set) and let $j \in I$ then define $X_i^j = X_i$ if $i = j$ and $X_i^j = \varnothing$ otherwise and let $\tau^j : X_\bullet^j \to X_\bullet$ be the obvious cartesian transformation. Then the colimit is $X_j \to X$ and effectivity of descent shows that the bottom right square is a pullback for all $i$.

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_j \arrow[d] \\
A_i \arrow[r] & \coprod A_\bullet
\end{tikzcd}$$

for the converse and other parts see [AJ19]. ☐

**3.59.** A key notion bridging the gap between topoi and logoi is that of presentability. Recall from 3.20 that an category is presentable if it is a reflexive subcategory of some category of presheaves. An **topos** by itself is just a presentable logos. A logos morphism between presentable logoi is then a cocontinuous map between presentable categories and so has a right adjoint. Then a morphism of topoi is such a right adjoint: an functor with a left adjoint preserving finite limits. We can sum up this situation by saying that the category of topoi is the opposite of the category of presentable logoi.

**topos**

**3.60.** In the presence of the presentability assumption there is an alternative axiomatization of logoi due to Lawvere and Tierney. We begin by yet another reformulation of the universality of colimits. After that we note that effectivity of descent for restricted descent data can be formulated using classifying objects.

**3.61.** *An locally cartesian closed category has universal colimits, if the category in question is presentable the converse holds.*

*Proof.* The first direction is trivial. Suppose that $\mathcal{E}$ is presentable then so are all it's slices $A \mapsto \mathcal{E}_{/A}$. Since $\mathcal{E}$ has finite limits the assignment above becomes an functor sending a map $f : A \to B$ to the change of base functor $f^* : \mathcal{E}_{/B} \to \mathcal{E}_{/A}$. By assumption this map preserves colimits. By the adjoint functor theorem for presentable categories this map has a right adjoint so $\mathcal{E}$ is locally cartesian closed. ☐

**3.62.** Let $S$ be a class of morphisms in a category $\mathcal{E}$ and $X_\bullet$ an diagram in $\mathcal{E}$. Then the **category of** $S$ **descent data** $\mathsf{Desc}(X_\bullet, S)$ is the category of cartesian natural transformations $Y_\bullet \Rightarrow X_\bullet$ such that each component is in $S$. Similarly we let $\mathcal{E}_{/X}|_S$ be the category of all morphisms in $S$ with codomain $X$ in $\mathcal{E}$.

**category of** $S$ **descent data**

**3.63.** The adjunction above restricts to $S$ if we require that $S$ is closed under pullbacks

$$\begin{tikzcd} \mathcal{E}_{/X}|_S \arrow[r, bend right, "\mathsf{dec}_{X_\bullet}"'] \arrow[r, phantom, "\perp"] & \mathsf{Desc}(X_\bullet, S) \arrow[l, bend right, "\mathsf{glue}_{X_\bullet}"'] \end{tikzcd}$$

In this case we can speak of faithful/effective descent for $S$ morphisms.

**3.64.** Let $\mathcal{E}$ be a category with pullbacks. Then a $S$-**classifier** for a class of morphisms $S$ is an object $\mathcal{U}_S$ with a map $p_S : \widetilde{\mathcal{U}}_S \to \mathcal{U}_S$ in $S$ such that the functor $X \mapsto \mathcal{E}_{/X}|_S$ is represented by $\mathsf{Hom}(-, \mathcal{U}_S)$. In other words $(p, \mathcal{U}_S)$ is the terminal object in the category of elements $\int_{X \in \mathcal{E}} \mathcal{E}_{/X}|_S$. This means that isomorphism classes of $S$-map $f : Y \to X$ over $X$ correspond bijectively with maps $\chi_f : \mathrm{X} \to \mathcal{U}_s$ and $\chi_f^* p$ is isomorphic to $f$.

$S$-**classifier**

**3.65.** *Let $\mathcal{E}$ be an lex cc category and let $S$ be a pullback stable class of maps. If $\mathcal{E}$ has an $S$-classifier then it has effective descent for $S$ morphisms.*

*Proof.* To show that we have effective descent, suppose that we have a $S$-cartesian natural transformation $f_i : Y_i \to X_i$. Then by assumption there are maps $\chi_i : X_i \to \mathcal{U}_S$ such that $\chi_i^* p \cong f$. These maps form a $X_\bullet$ cocone over $\mathcal{U}_S$ and so induce a unique map $\chi : X \to \mathcal{U}_S$ which yield an $S$-map $f : Z \to X$ displayed below left which is the putative colimit. In the diagram below all squares a pullback squares.

$$\begin{tikzcd}
& Y_2 \arrow[dd, "f_2"] \arrow[drr, bend left] & & \\
Y_1 \arrow[ur] \arrow[dd, "f_1"] \arrow[rrr, bend right] & & Z \arrow[r] \arrow[dd, "f"] & \widetilde{\mathcal{U}}_S \arrow[dd, "p_S"] \\
& X_2 \arrow[dr] \arrow[drr, bend left, "\chi_2"] & & \\
X_1 \arrow[ur] \arrow[rr] \arrow[rrr, bend right, "\chi_1"] & & X \arrow[r, dashed, "\chi"] & \mathcal{U}_S
\end{tikzcd}$$

If we decompose $f : Z \to X$ along the legs $X_i \to X$ we get $Z_i \to X_i$. By the pullback pasting law these are isomorphic to $f_i : Y_i \to X_i$. And by universality of colimits they glue back into Z. In other words $Z$ is isomorphic to the colimit of $Y_\bullet$. □

**3.66.** A category with finite limits has an **subobject classifier** $\Omega$ if it has an $S_{\mathsf{mono}}$-classifier, where $S_{\mathsf{mono}}$ is the class of monomorphisms. NB: Monomorphisms are always pullback stable.

**subobject classifier**

**3.67.** The universal subobject is a map out of the terminal object $\mathbf{1}$ traditionally called $\mathsf{true} : \mathbf{1} \to \Omega$.

**3.68.** The Lawvere-Tierney conception of a logos, or an **elementary logos**, is an locally cartesian closed category with a subobject classifier.

**elementary logos**

**3.69.** Under presentability the Lawver-Tierney definition is equivalent to the earlier definitions, see [AJ19] and [MM12].

**3.70.** There is one final definition of a logos which we wish to give. Recall that a presentable category is an reflective localization of an presheaf category. Then an presentable category is an logos if the reflector $R$ is left exact. In fact the converse also holds, presentable logoi are exactly left exact localizations of presheaf categories, but we will not show this.

**3.71.** *Let $\mathcal{S}$ be an lex cc category with a reflective subcategory $i : \mathcal{E} \to \mathcal{S}$ such that the reflector $R : \mathcal{S} \to \mathcal{E}$ is left exact. Then*

*(i) $X_\bullet$ is of effective colimit in $\mathcal{E}$ if $iX_\bullet$ is effective in $\mathcal{S}$.*

*(ii) $X_\bullet$ is of faithful descent in $\mathcal{E}$ if $iX_\bullet$ is of faithful descent in $\mathcal{S}$.*

*Proof.* The reflective subcategory $\mathcal{E}$ inherits all colimits and finite limits from $\mathcal{S}$ where the colimits are computed by applying the reflector (see 2.6). (i). Suppose we have an cartesian transformation $Y_\bullet \Rightarrow X_\bullet$ in $\mathcal{E}$ then this is also a cartesian transformation in $\mathcal{S}$ because $i$ preserves limits. Then if colimits for $iX_\bullet$ have effective descent in $\mathcal{S}$ this extends to an cartesian transformation of colimit diagrams $iY_\bullet^{\triangleright} \Rightarrow iX_\bullet^{\triangleright}$. Applying the reflector yields an cartesian diagram with parts as displayed below left. But because $i \dashv R$ is an reflective subcategory $Ri \cong \mathsf{id}$ and the reflector of the colimit is just the colimit $R(\mathsf{colim}_i\, iY_i) = \mathsf{colim}_i\, Y_i$. So then $X_i$ is an effective colimit in $\mathcal{E}$.

$$\begin{tikzcd}
Ri(Y_i) \arrow[r] \arrow[d] & R(\mathsf{colim}_i\, iY_i) \arrow[d] \\
Ri(X_i) \arrow[r] & R(\mathsf{colim}_i\, iX_i)
\end{tikzcd}
\qquad
\begin{tikzcd}
iY_i \arrow[rr] \arrow[dr, dashed, "\ulcorner"{pos=0.2}] \arrow[dd] & & iY \arrow[dd] \\
& Z \arrow[ur] \arrow[dd] & \\
iX_i \arrow[rr] \arrow[dr] & & iX_\infty \\
& \mathsf{colim}_i\, iX_i \arrow[ur, dashed, "\phi"] &
\end{tikzcd}$$

(ii). Suppose that we have an diagram $X_\bullet$ with colimit $X_\infty = \mathsf{colim}_i\, X_i$ and a map $f : Y \to X_\infty$, by decomposition (pullback) we get an diagram $Y_i = Y \times_X X_i$. Then under the inclusion $i$ into $\mathcal{S}$ this becomes the diagram displayed above right were the far square is a pullback. We form the colimit of $iX_\bullet$ in $\mathcal{S}$ which induces a map into $iX_\infty$. We take the pullback along this map yielding $Z$. By the universal property of the pullback we get maps $iY_i \to Z$. By the pullback pasting law all triangles commute. This means we have an cartesian transformation in $\mathcal{S}$ over the colimit of $iX_\bullet$, now if $iX_i$ colimits are of faithful descent we get $Z = \mathsf{colim}_i\, iY_i$. After reflecting back into $\mathcal{E}$ the map $\phi$ turns into the identity but since $R$ preserves colimits this means that $R(\mathsf{colim}_i\, iY_i) = R(Z) = Ri(Y) = Y$ and so $Y$ is the colimit. □

**3.72.** *Suppose that $\mathcal{S}$ is an logos and let $\mathcal{E}$ be an left exact reflective localization, then $\mathcal{E}$ is also an logos.*

The above is easily seen by apply the preceding lemma 3.71. What is now left to show is that presheaf categories are logoi.

**3.73.** *The following are logoi*

*(i) The category $\mathcal{S}\mathrm{et}$ of sets.*

*(ii) For any small category $\mathsf{C}$ the category of presheaves $\mathcal{P}\mathit{sh}(\mathsf{C})$.*

*Proof.* (i) In $\mathcal{S}\mathrm{et}$ colimits are universal because it is locally cartesian closed. Then it enough to show disjointness of sums and effectivity of equivalence relations. These are both standard facts (see [MM12, Appendix]).

(ii) Suppose that $X_\bullet$ and $Y_\bullet$ are $\mathsf{I}^\triangleright$ diagrams in $\mathcal{P}\mathrm{sh}(\mathsf{C})$ with a natural transformation $\alpha : Y_\bullet \Rightarrow X_\bullet$ restricting to an cartesian transformation on $\mathsf{I}$ and that $X_\bullet$ is a colimit diagram. Then for each $c \in \mathsf{C}$ there is a transformation $\alpha(c) : Y_\bullet(c) \Rightarrow X_\bullet(C)$ in $\mathcal{S}\mathrm{et}$. Since (co)limits are computed pointwise we get $X_\infty(c) = \mathsf{colim}_i\, X_i(c)$ and $\alpha(c)$ also restricts to an cartesian transformation. We conclude that $Y_\infty(c) = \mathsf{colim}_i\, Y_i(c)$ iff $\alpha(c)$ is cartesian. But $\alpha(c)$'s are all cartesian iff $\alpha$ is and for all $Y_\infty(c) = \mathsf{colim}_i\, Y_i(c)$ iff $Y_\infty = \mathsf{colim}_i\, Y_i$. In particular the presheaf category inherits universal limits and effective descent of homotopically discrete colimits from $\mathcal{S}\mathrm{et}$. □

# Part II

# Homotopy mathematics

Let's go back to Klein's erlangen program from the introduction. Remember that he proposes to study mathematical objects using ambient models equiped with a group of symmetries under which our statement are to be invariant. The principle deficiency with associating only a group to capture invariants is that the objects might have much richer collection of symmetries. More precisely, there might be multiple non equal symmetries with the same beginning and end points. These symmetries can then be potentially connected with higher symmetries which do not have to be unique and so on. This lead to a notion of an $\infty$-group or, slightly generalized, to an $\infty$-groupoid. We cite Shulman:

> This notion may seem very abstruse, but over the past few decades $\infty$-groupoids have risen to a central role in mathematics and even physics, starting from algebraic topology and metastasizing outwards into commutative algebra, algebraic geometry, differential geometry, gauge field theory, computer science, logic, and even combinatorics. It turns out to be very common that two things can be equal in more than one way.

An paradigmic example of objects with higher identifications is the homotopy theory of spaces which we will recall in chapter 4. To capture the 'symmetries' of topological spaces we introduce the theory of homotopical categories in chapter 5. Since these symmetries do not form a simple group it will not be enough to simply quotient the category by these symmetries. Instead we have to work with the category and it's symmetries directly. If topological spaces with their weak equivalences fit in the picture described above they are the analytic model of some theory. Conjecturally this underlying theory is precisely the theory of $\infty$-groupoids. Describing this theory directly, i.e. synthetically, is in a sense outside of the power of 20th century algebraic methods. Actually it is believed that these methods do exists and that they are given by homotopy type theory. However this synthetic theory still lacks an satisfactory justification in ordinary set based mathematics[2]. Consequently the only way of working with this theory is through the models. To aid this program we introduce in chapter 6 simplicial sets which provide another model of the underlying theory.

The category theory we studied in part I has an intimate connection to sets. We will see in chapter 7 that ordinary category theory can be generalized to enriched category theory. We then immediately move on to considering categories enriched in $\infty$-groupoids which are supposed to be $(\infty, 1)$-categories. If ordinary categories organize models of algebraic theories then $(\infty, 1)$-categories organize models of theories under an $\infty$-groupoid of symmetries. Of course, just as with $\infty$-groupoids, there is no clear notion of an $(\infty, 1)$-category and we instead have to study them through models which we will introduce in 7.2.

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[2] Recently Mike Shulman provided a crucial piece of this puzzle, see [Shu19]

# Chapter 4

# Classical homotopy theory

In the category of topological spaces the notion of isomorphism is inadequate to capture our intuition about which spaces should be somehow 'equivalent'. In particular, a space $A$ and its 'thickening' $X$, such as a point $A = \{x \in \mathbb{R}^2 \mid |x| = 0\}$ and a small disk $X = \{x \in \mathbb{R}^2 \mid |x| \leq \epsilon\}$ should be equivalent, but there is no homeomorphism between them. We can however envision making the disk $X$ smaller and smaller in a time span $t \in [0,1]$ until at $t = 1$ the disk is squeezed down to the point $A$. The interval $[0,1]$ will also be written $I$ and plays a vital role in homotopy theory.

This idea is captured by the notion of a **deformation retract** from a space $X$ onto the subspace $A \subset X$. Such a deformation retract is a family of maps $f_t : X \to X$ such that $f_0 = \mathsf{id}_X$ and $f_1(X) = A$ which is continuous as a map $[0,1] \times X \to X, (t,x) \mapsto f_t(x)$. If there is a deformation retract of a space $X$ onto $A$ we say that $A$ is a deformation retract of $X$ and write $X \rightsquigarrow A$. To serve as a notion of equivalence on spaces we would need that $X \rightsquigarrow A$ is an equivalence relation, it is however only reflexive and transitive. Fortunately enough, the symmetric closure of $\rightsquigarrow$ is easy enough to describe explicitly. In order to do this, we will need the central notion of homotopy.

**deformation retract**

**4.1.** A **(left) homotopy** is a family of maps $f_t : X \to Y$ with $t \in [0,1]$ such that $I \times X \to Y, (t,x) \mapsto f_t(x)$ is continuous. We say two maps $g, h : X \to Y$ are **homotopic** and write $g \sim h$ if there is a homotopy $f_t$ with $g = f_0$ and $h = f_1$. Now a map between $f : A \to B$ is a **homotopy equivalence** if there is a $g : B \to A$ such that $fg \sim \mathsf{id}$ and $gf \sim \mathsf{id}$. If two spaces $A$ and $B$ are connected by a homotopy equivalence they are said to be **homotopy equivalent**, we write $A \simeq B$.

**(left) homotopy**

**homotopy equivalence**

**4.2.** Note that $A$ is a deformation retract of $X$ iff the inclusion $i : A \to X$ is an homotopy equivalence. In fact homotopy equivalence is in the symmetric closure of deformation retracts: For any $A \simeq B$ there is a space $C$ such that $A \leftsquigarrow C \rightsquigarrow B$ (see [Hat00]).

**4.3.** The structure of the interval $I$ shows that homotopy is an equivalence relation on $\mathsf{Hom}(X,Y)$ the set of maps between $X$ and $Y$. Consequently we can form $\mathsf{Hom}(X,Y)_{/\sim}$ the set of homotopy classes of maps.

**4.4.** Suppose we have a left homotopy $H : X \times I \to Y$. Fixing a point $x \in X$ we obtain a mapping $H_x : t \mapsto H(x,t)$ to paths $I \to Y$. The space of such maps written $Y^I$ can be topologized using the compact open topology, with this the assignment $x \mapsto H_x$ becomes continuous.

$$\begin{tikzcd}
\mathsf{Hom}(X \times I, Y) \arrow[r, "\sim"] & \mathsf{Hom}(X, Y^I) \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
H \arrow[r, mapsto] & x \mapsto H_x
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times I \arrow[r, "(f{,}\mathsf{id})"] & Y^I \times I \arrow[r, "\epsilon"] & Y
\end{tikzcd}$$

Since the space $I$ is locally compact the evaluation map $\epsilon : I \times Y^I \to Y$ is continuous. Using this we get an inverse to the mapping above, sending a map $h : X \to Y^I$ to the composite displayed above right. This implies that a map $h : X \to Y^I$ is 'just as good' as a homotopy $H : X \times I \to Y$, in particular there is an bijection between such maps displayed above left.

**4.5.** A **(right) homotopy** from $g$ to $g'$ is a map $h : X \to Y^I$ such that for each $x$ the path $\epsilon(h(x), \mathsf{id}_I) : I \to Y$ is a path from $g(x)$ to $g'(x)$. We will often identify an element of $Y^I$ with the map $I \to X$ that it represents, with this we can just write that $h(x)$ is a path between $g(x)$ and $g'(x)$.

**4.6.** *For any morphism $f : A \to B$ the natural action by pre- and postcomposition of morphisms respects $\sim$.*

*Proof.* That post composition respects $\sim$ is obvious since an homotopy $g \sim g' : X \to A$ is represented by a map $I \times X \to A$. A left homotopy $f \sim f' : B \to Y$ can be equivalently presented by a right homotopy $h : B \to Y^I$ and so we get that postcomposition also respects $\sim$. ☐

**4.7.** By the above discussion we can form the **naive homotopy category of spaces** $h\mathcal{T}\mathrm{op}$ with objects the same objects as $\mathcal{T}\mathrm{op}$ such that $\mathsf{Hom}_{h\mathcal{T}\mathrm{op}}(A,B) := \mathsf{Hom}(A,B)_{/\sim}$.

**4.8.** *Homotopy equivalence is an equivalence relation on the objects of $\mathcal{T}\mathrm{op}$.*

*Proof.* Reflexivity and symmetry are trivial. For transitivity, if $A \simeq B$ and $B \simeq C$ by the equivalences displayed below

$$\begin{tikzcd}
A \arrow[r, shift left, "f_+"] & B \arrow[l, shift left, "f_-"] \arrow[r, shift left, "g_+"] & C \arrow[l, shift left, "g_-"]
\end{tikzcd}$$

then since composition respects $\sim$ we have $g_+f_+f_-g_- \sim g_+\mathsf{id}_B g_- \sim g_+g_- \sim \mathsf{id}_C$ and $f_-g_-g_+f_+ \sim f_-\mathsf{id}_B f_+ \sim f_-f_+ \sim \mathsf{id}_A$ so $A \simeq C$. ☐

**4.9.** There is an functor $\Gamma : \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$ sending a space to its underlying set of points. This functor is represented by $\mathbf{1}$, the space consisting of just a point, so that $\Gamma(X) = \mathsf{Hom}(\mathbf{1}, X)$.

**4.10.** The functor $\Gamma : \mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}$ has both a left and a right adjoint $\mathsf{Disc} \dashv \Gamma \dashv \mathsf{coDisc}$ which represent the minimal and maximal ways of topologizing a space. Concretely $\mathsf{Disc}(S)$ equips a set $S$ with the **discrete topology** where the topology is $\mathcal{P}(S)$. Dually $\mathsf{coDisc}(S)$ equips $S$ with the **codiscrete topology** given $\{\varnothing, S\}$, i.e. all points are close together. We

**discrete topology**

will sometimes pretend that a set is a space, then we will always consider it equipped with the discrete topology.

## 4.1 Homotopy groups and weak equivalences

With this new notion of equivalence we might ask if one can classify all spaces up to homotopy, or at least find invariants to tell us when spaces are not homotopy equivalent. The most important such invariant are the homotopy groupoids $\pi_n(X)$ for $n \geq 0$ of a space $X$. We will first define the collection of path components of a space $X$. Then the homotopy groupoids will be defined using the notion of path components of a space.

**4.11.** Given two points $x, y \in X$ a homotopy between the induced $x, y : \mathbf{1} \to X$ is a **path**, this is a map $p : I \to Y$ such that $p(0) = x$ and $p(1) = y$. If there is such a path we write $x \overset{\text{path}}{\sim} y$, which is just the homotopy relation $\sim$ for points. Since $\sim$ is an equivalence relation, so is $\overset{\text{path}}{\sim}$ and we can form the quotient $\pi_0(X) := X/\overset{\text{path}}{\sim}$ of **path connected components** of the space $X$.

**path**

**path connected components**

**4.12.** In order to define the higher homotopy groupoids of a space it it useful to define the category of relative spaces $\mathcal{T}\mathrm{op}_{\mathsf{rel}}$.

- A **relative space** $(X, X_0)$ is a space $X \in \mathcal{T}\mathrm{op}$ with a subspace $X_0 \subset X$.
- A **relative map** between relative spaces $f : (X, X_0) \to (Y, Y_0)$ is a map $f : X \to Y$ such that $f(X_0) \subset Y_0$. Write $\mathsf{Hom}(X, X_0; Y, Y_0)$ for the set of relative maps between $(X, X_0)$ and $(Y, Y_0)$
- A **relative homotopy** between $f, g : (X, X_0) \to (Y, Y_0)$ is an homotopy $h_t : X \to Y$ such that $h_t$ is a relative map $(X, X_0) \to (Y, Y_0)$ for all $t$, in this case we will also write $f \sim g$.

**4.13.** A relative space $X_0 \subset X$ such that $X_0$ is just a point $\{x_0\}$ is called a **pointed space** and $x_0$ is called the basepoint. The full category spanned by the pointed spaces is called the category of **pointed spaces** $\mathcal{T}\mathrm{op}_*$. In fact $\mathcal{T}\mathrm{op}_* \hookrightarrow \mathcal{T}\mathrm{op}$ is a reflexive subcategory where the reflector $L$ sends $X \mapsto (X \sqcup \{*\}, \{*\})$ with the coproduct topology.

**pointed space**

**4.14.** Let $\mathsf{Hom}(X, X_0; Y, Y_0)_{/\sim}$ be the set of relative maps $\mathsf{Hom}(X, X_0; Y, Y_0)$ quotiented by the relative homotopy relation, this is also a functorial with respect to pre- and postcomposition for the same reason the non relative version is.

The final requirement for defining the higher homotopy groupoids will be the higher analogous of the interval $I$. Just as the path components of space are determined by the functor $\mathsf{Hom}(I, -)$ so will the higher homotopy groupoids be determined by maps $\mathsf{Hom}(I^n, -)$ where $I^n$ are defined as follows.

**4.15.** For each $n \geq 0$ let $I^n$ be the $n$ fold cartesian product of the standard interval $[0, 1] \in \mathcal{T}\mathrm{op}$. Let the boundary $\partial I^n$ for $n > 0$ be the topological boundary of this space, for $n = 0$ let $\partial I^0 := \varnothing$. In both cases explicitly given by those $\mathbf{x} \in I^n$ with at least one coordinate $0 < i \leq n$ such that $x_i \in \{0, 1\}$.

Other important spaces for homotopy theory are the $n$-spheres $S^n$ and $n$-disks $D^n$ defined below. Note that $D^0 = \{0\}$ and $S^{-1} = \varnothing$, so we have $I^n \cong D^n$ and $\partial I^n \cong S^{n-1}$.

$$S^{n-1} := \{x \in \mathbb{R}^n \mid |x| = 1\}, \qquad D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}, \qquad \text{for } n \geq 0$$

**4.16.** *If $x, y \in \pi_0(X)$ and there is an isomorphism $\mathsf{Hom}(I^n, \partial I^n; X, x)_{/\sim} \cong \mathsf{Hom}(I^n, \partial I^n; X, y)_{/\sim}$ for $n \geq 1$.*

*Proof.* Full proof in [Hat00, Section 4.1] □

**4.17.** For a pointed space $(X, x)$ the $n$**th homotopy group** $\pi_n(X, x)$ is the homotopy equivalence classes of pointed maps $(I^n, \partial I^n) \to (X, x)$.

**$n$th homotopy group**

The $n$**th homotopy groupoid** of a space $X$ is a category $\pi_n(X)$ with an object $[x]$ for each $[x] \in \pi_0(x)$, and for each $[x], [y] \in \pi_0(X)$ let

**$n$th homotopy groupoid**

$$\mathsf{Hom}([x],[y]) = \begin{cases} \varnothing & \text{if } [x] \neq [y] \\ \pi_n(X, x) & \text{if } [x] = [y] \end{cases}$$

In other words $\pi_n(X)$ records for every path connected component $[x] \in \pi_0(X)$ the homotopy group $\pi_n(X, x)$. Theorem 4.16 shows that the homotopy groupoids are well defined. The notation $\pi_0(X)$ represents both the set of path components and the 0th homotopy groupoid, this notation is consistent since $\pi_0(X)$ is discrete as a groupoid.

**4.18.** If the space $X$ is path connected the notions of $n$th homotopy groupoid and $n$th homotopy group (choosing any basepoint) coincide. Since a space can always be decomposed into its path components and all homotopical notions can then be defined 'path component wise' it is sufficient to study path connected spaces. This is why historically there was more emphasis on homotopy groups and not homotopy groupoids.

The homotopy groupoids of a space are very powerful invariants used to answer many practical questions about spaces. They however fail to detect all homotopy equivalent spaces, usually this is due to singularities where lines come arbitrarily close without forming a path. Since these spaces are often seen as pathologies this failure is not much of an issue in practice. Instead the homotopy groupoids induce their own notion of equivalence which often supersedes the notion of homotopy equivalence.

**4.19.** A map $f : X \to Y$ between spaces is a **weak (homotopy) equivalence** if the induced $\pi_n(X) \to \pi_n(Y)$ is an equivalence of groupoids for each $n \in \mathbb{N}$.

**weak (homotopy) equivalence**

**4.20.** The homotopy groupoids can be packaged up into one functor $\pi : \mathcal{T}\mathrm{op} \to \mathcal{G}\mathrm{pd}^{\mathbb{N}}$ where $\mathbb{N}$ is discrete. This assigning to a space $X$ all of its fundamental groupoids $\{\pi_n(X)\}$. Two objects $F$ and $G$ of $\mathcal{G}\mathrm{pd}^{\mathbb{N}}$ and $G$ are equivalent if for each $n \in \mathbb{N}$ the groupoid $F(n) \simeq G(n)$. This means that the kernel $\mathsf{ker}(\pi)$ are precisely the weak equivalences.

## 4.2 CW-complexes, fibrations and cofibrations

Modern homotopy theory is often the study of spaces up to weak equivalence, or, perhaps more accurately, the study of spaces where weak equivalence and homotopy equivalence coincides. These spaces are the CW-complexes which we will define now.

**4.21.** Suppose we have a space $X$ and a map $\phi : S^{n-1} \to X$ we can **attach an $n$-cell along $\phi$ to** $X$ using the pushout displayed below left with the boundary inclusion $S^{n-1} \hookrightarrow D^n$.

$$\begin{tikzcd} S^{n-1} \arrow[r, "\phi"] \arrow[d, hook, "i"] & X \arrow[d] \\ D^n \arrow[r] & X' \end{tikzcd} \qquad \begin{tikzcd} \sum_{i\in I} S^{n-1} \arrow[r, "{[\phi_i]_{i\in I}}"] \arrow[d, hook, "\sum_{i\in I} i"] & X \arrow[d] \\ \sum_{i\in I} D^n \arrow[r] & X' \end{tikzcd}$$

In fact we can attach multiple such $n$-cells at once. A set $I$ of attaching maps $\phi_i : S^{n-1} \to X$ for $i \in I$ can be collected using the coproduct $[\phi_i]_{i\in I} : \sum_{i\in I} S^{n-1} \to X$, then attached using the pushout displayed above right.

**4.22.** A **CW-complex** or **cell complex** is a space $X$ that can be obtained by the following method: Start with the empty space $X_{-1} := \varnothing$. Suppose $X_{n-1}$ is constructed. Then there is a set $J_n$ of attaching maps $\phi_i : S^{n-1} \to X_{n-1}$ for $n$-cells onto $X_{n-1}$ for each $i \in J_n$. The space $X_n$ is obtained by attaching the $n$-cells as displayed above and comes with inclusion $i : X_{n-1} \to X_n$. Then $X$ is the union of all $X_n$, or in more categorical terms $X = \mathsf{colim}_{n\in\mathbb{N}} X_n$.

**CW-complex**

Intuitively the $n$th homotopy groupoid of a space $X$ characterizes the maps $I^n \to X$ such that $\partial I^n \mapsto \{x\}$ some fixed $x \in X$ up to homotopy. Since $I^n \cong D^n$ and $\partial I^n \cong S^{n-1}$ we see a close relation with the CW-complexes: in a sense a CW-complex consists only of such maps. Therefore it is reasonable to suspect that the homotopy groupoids can 'detect all there is to know' about a CW-complex up to homotopy. Indeed this is the context of the following theorem.

**4.23.** *For CW-complexes $X$ and $Y$ if there is an weak equivalence $f : X \to Y$ (i.e. a map such that $\pi(f)$ is an isomorphism) then $f$ is an homotopy equivalence.*

**Whitehead theorem**

*Proof.* See Hatcher [Hat00] ☐

**4.24.** The map $\partial : \{0\} \hookrightarrow I$ including an endpoint into the interval is an homotopy equivalence by $\sigma : I \to \{0\}$. This map determines the following structure on the category $\mathcal{T}\mathrm{op}$.

- For any space $Y$ there is a homotopy equivalence $Y^{\partial} : Y^I \to Y^{\{0\}} \cong Y$. The collection of these maps are called the **generating trivial Hurewicz fibrations**.
- For any space $X$ there is a homotopy equivalence $A \times \partial : A \cong A \times \{0\} \to A \times I$. The collection of these maps are called the **generating trivial Hurewicz cofibrations**.

**4.25.** A **Hurewicz cofibration** is a map $i : A \to X$ having the right lifting property against the generating trivial Hurewicz fibrations. Dually, a **Hurewicz fibration** is a map $p : E \to B$ having the left lifting property against the generating trivial Hurewicz cofibrations.

**Hurewicz cofibration**

**Hurewicz fibration**

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "i"] & Y^I \arrow[d, "Y^\partial"] \\ X \arrow[r, "F_0"] \arrow[ur, dotted, "\tilde{F}"] & Y^{\{0\}} \end{tikzcd} \qquad \begin{tikzcd} A \times \{0\} \arrow[r, "F_0"] \arrow[d, "A\times\partial"] & E \arrow[d, "p"] \\ A \times I \arrow[r, "F"] \arrow[ur, dotted, "\tilde{F}"] & B \end{tikzcd}$$

In the above diagrams $F$ is a homotopy, $F_0$ is a extension of one end of the homotopy, and the lift $\tilde{F}$ extends $F$ and $F_0$.

**4.26.** From now on it will be good to restrict attention to a better behaved subcategory of $\mathcal{T}\mathrm{op}$. From now on we will work with the subcategory of compactly generated and weakly Hausdorff spaces which we will just call $\mathcal{T}\mathrm{op}$.

- A weakly Hausdorff space $X$ has the property that the image of a compact space $A$ under $i : A \to X$ is closed in $X$, indeed any Hausdorff space is weakly Hausdorff.
- A space $X$ is compactly generated if its topology is determined by its compact subspaces $K \subset X$. In particular $A \subset X$ is closed (open) iff $A$ is closed (open) in $K$ for all compact $K \subset X$.

This class of spaces contains nearly any non pathological space we care about, among them all topological manifolds and every CW-complex.

**4.27.** It it not hard to check that any homeomorphism (and so identity) is a fibration, a cofibration and a homotopy equivalence. Moreover the fibrations, cofibrations and homotopy equivalences are closed under composition, with this we have the following wide subcategories of $\mathcal{T}\mathrm{op}$:

- $\mathbf{h}\mathcal{E}\mathrm{q}$ of homotopy equivalences, an homotopy equivalence is drawn as $A \xrightarrow{\sim} B$
- $\mathbf{h}\mathcal{C}\mathrm{of}$ of cofibrations, an cofibration is drawn with hook $A \hookrightarrow B$
- $\mathbf{h}\mathcal{F}\mathrm{ib}$ of fibrations, an fibration is drawn with double head $A \twoheadrightarrow B$

**mapping cylinder**

**4.28.** For an map $f : A \to B$ the **mapping cylinder** of $f$ given by the pushout below left

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\partial\times A"] & B \arrow[d] \\ I \times A \arrow[r] & \mathsf{Cyl}(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Path}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "f"] \\ B^I \arrow[r, "B^\partial"] & B \end{tikzcd}$$

dually the **mapping path space** $\mathsf{Path}(f)$ is given by the pullback above right.

**mapping path space**

**4.29.** We get the following very pleasing duality. A map is called a trivial fibration if it is in $(\mathsf{C}_h)^{\boxslash}$, i.e. it has the right lifting property against the cofibrations. Similarly a map is called trivial cofibration if it is in ${}^{\boxslash}(\mathsf{F}_h)$. Then

- $(\mathsf{C}_h)^{\boxslash} = \mathsf{E}_h \cap \mathsf{F}_h$
- $\mathsf{E}_h \cap \mathsf{C}_h = {}^{\boxslash}(\mathsf{F}_h)$

this means that $(\mathsf{C}, \mathsf{E}_h \cap \mathsf{F}_h)$ and $(\mathsf{F}_h \cap \mathsf{E}_h, \mathsf{C})$ are weak factorization systems. In fact these factorizations are functorial and can be described explicitly. Every map $f : A \to B$ factors through the mapping cylinder and the mapping path space as displayed below.

$$\begin{tikzcd}
& \mathsf{Path}(f) \arrow[dr, two heads] & \\
A \arrow[ur, hook, "\sim"] \arrow[rr, "f"] \arrow[dr, hook] & & B \\
& \mathsf{Cyl}(f) \arrow[ur, two heads, "\sim"'] &
\end{tikzcd}$$

**4.30.** Note: for this duality to work to work the restriction $\mathcal{T}\mathrm{op}$ to weakly Hausdorff compactly generated spaces is essentially. Indeed without it the cofibrations need to be replaced with closed cofibrations, i.e cofibrations $i : A \to X$ such that $i(A)$ is closed in $X$. In our $\mathcal{T}\mathrm{op}$ every cofibration is closed due to the compactly generated property.

# Chapter 5

# Homotopical category theory

Topological considerations lead us to define the homotopy relation $\sim$ between maps in $\mathcal{T}\mathrm{op}$. Intuitively we would like to 'quotient' the morphisms of $\mathcal{T}\mathrm{op}$ by this relation. One can do this to obtain the naive homotopy category $h\mathcal{T}\mathrm{op}$ with the objects of $\mathcal{T}\mathrm{op}$ and homotopy classes of maps $\mathsf{Hom}_{h\mathcal{T}\mathrm{op}}(X, Y)$. In this chapter we will give an more abstract category theoretical investigation of this situation.

## 5.1 Categories with a homotopy relation

**homotopy relation**

**5.1.** A **homotopy relation** $\sim$ on a category $\mathsf{C}$ consists of an equivalence relation on the morphisms of $\mathsf{C}$ such that $f \sim g$ implies that $f$ and $g$ are parallel arrows in $\mathsf{C}$ and such that $\sim$ respects composition

- If $f \sim g$ with $f, g : A \to B$ and $h : B \to C$ then $hf \sim hg$.
- If $f \sim g$ with $f, g : B \to C$ and $h : A \to B$ then $fh \sim gh$.

**native homotopy category**

**5.2.** Given a category $\mathsf{C}$ with a homotopy relation $\sim$ the **native homotopy category** $h\mathsf{C}$ is the category with the same objects as $\mathsf{C}$ but where the morphisms are quotiented by the homotopy relation. One easily checks that this is a category. This construction comes with a bijective-on-objects quotient map $\gamma : \mathsf{C} \to h\mathsf{C}$. The category $h\mathsf{C}$ together with the map $\gamma : \mathsf{C} \to h\mathsf{C}$ is called the homotopy localization of $\mathsf{C}$ by $\sim$, usually $\sim$ is understood from context. The homotopy class of $f$, i.e. its image under $\gamma$, is also written $[f]$.

**5.3.** The above construction enjoys the following universal property, let $\mathsf{C}$ be a category with homotopy relation $\sim$ then for every functor $F$ such that $f \sim g \to F(f) = F(g)$ there is a unique $F_!$ rendering the diagram below commutative.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathsf{D} \\
h\mathsf{C} \arrow[ur, dashed, "F_!"'] &
\end{tikzcd}$$

**5.4.** Every functor $F : \mathsf{C} \to \mathsf{D}$ yields an **induced homotopy relation** $\sim_F$ on $\mathsf{C}$ where $f \sim_F g$ iff $F(f) = F(g)$. Functionality and the properties of composition in $\mathsf{D}$ ensure that this is indeed an homotopy relation.

**induced homotopy relation**

**5.5.** Consider a reflective subcategory $i : \mathsf{C} \hookrightarrow \mathcal{E}$ with reflector $R : \mathcal{E} \to \mathsf{C}$, then consider the naive homootopy category $h\mathcal{E}$ obtained by the induced homotopy relation $\sim_R$. By the universal property we obtain a map $R_! : h\mathcal{E} \to \mathsf{C}$ such that $R_!\gamma = R$. The functor $R_!$ is

- essentially surjective: any object $c \in \mathsf{C}$ produces the object $\gamma i(c) \in h\mathcal{E}$ and $R_!\gamma i(c) = Ri(c) = c$.
- faithful: given $f$ in $\mathsf{C}$ the map $\gamma i(f)$ in $h\mathcal{E}$ satisfies $R_!\gamma i(f) = Ri(f) = f$.
- full: if $[f] \neq [g]$ in $h\mathcal{E}$ then by definition $R(f) \neq R(g)$ and so $R_!([f]) = R(f) \neq R(g) = R_!([g])$.

So a reflective subcategory is equivalent to the naive homotopy category obtained from the induced homotopy relation of its reflector.

**5.6.** Consider a convenient category of topological spaces $\mathcal{T}\mathrm{op}$ such that right homotopy and left homotopy coincide. Left homotopy between of maps $X \to Y$ is represented by $I \times X \to Y$ so left homotopy preserves postcomposition. Similarly right homotopy preserves precomposition. Write $\sim_h$ for left/right homotopy between maps then this is a homotopy relation in the sense of this chapter and so $h\mathcal{T}\mathrm{op}$ is a homotopy localization.

**5.7.** Consider a category $\mathsf{C}$ with a homotopy relation $\sim$, we say that $f : A \to B$ is an **homotopy equivalence** if the following equivalent properties hold

**homotopy equivalence**

(i) The homotopy class $[f]$ is an isomorphism in $h\mathsf{C}$

(ii) There is a $g$ in $\mathsf{C}$ such that $fg \sim \mathsf{id}$ and $gf \sim \mathsf{id}$.

*Proof of equivalence.* If $[f]$ is an isomorphism then it has an inverse $[f]^{-1}$ which is a homotopy class of maps in $\mathsf{C}$ pick a representative $g$ such that $[g] = f^{-1}$ then this is its homotopy inverse. Conversely $fg \sim \mathsf{id}$ and $gf \sim \mathsf{id}$ means that $[f][g] = \mathsf{id}$ and $[g][f] = \mathsf{id}$, hence $[f]$ is an isomorphism. ☐

**5.8.** Every category $\mathsf{C}$ can be given the minimal homotopy relation where $f \sim g$ iff $f = g$. This is the homotopy relation induced by the identity functor and in particular $h\mathsf{C} \cong \mathsf{C}$.

**5.9.** Consider two categories $\mathsf{C}$ and $\mathsf{D}$ and a functor $F : \mathsf{C} \to \mathsf{D}$ then $F$ is said to be homotopy invariant if the following equivalent properties hold

- Given $f \sim g$ in $\mathsf{C}$ then $F(f) \sim F(g)$ in $\mathsf{D}$
- The composition $\gamma_D F$ factors through $h\mathsf{C}$ as displayed below.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "F"] \arrow[d, "\gamma_{\mathsf{C}}"'] & \mathsf{D} \arrow[d, "\gamma_{\mathsf{D}}"] \\
h\mathsf{C} \arrow[r, dotted] & h\mathsf{D}
\end{tikzcd}$$

*Proof.* Clearly if (i) holds then $\gamma_D F$ sends homotopic maps to identical maps so by the universal property of $h\mathsf{C}$ it factors as required by (ii). Conversely if (ii) and we are given $f \sim g$ then $[f] = [g]$ in $h\mathsf{C}$ and so $\gamma_{\mathsf{D}} F(f) = F_!([f]) = F_!([g]) = \gamma_{\mathsf{D}} F(g)$ but then $F(f) \sim F(g)$. □

**5.10.** In this sense, the homotopy category with the homotopy localizers can be seen as a convenient way to detect if functors are homotopy invariant. This should be compared to the way the homotopy groups of a space can be used to detect if a map is a weak homotopy equivalence. However, just as a a map between the homotopy groups of a space does not determine a map between the spaces, a map between homotopy categories also does not determine homotopy invariant maps.

## 5.2 Homotopical categories

In a sense homotopy invariant functors do not capture all situations where we want to describe invariant constructions. What we often really want is to replace 'equivalent' objects. Clearly every functor invariant under homotopy equivalence is invariant under changing homotopy equivalent objects. The problem is that sometimes we have a notion of equivalence that does not arise out of a homotopy relation. This usually means that we have a functor into a category of invariants where we decide equivalences in. In this case we work with a category and a wide subcategory of weak equivalences that 'behave like the isomorphisms of a category'.

**5.11.** A **homotopical category** is a category $\mathsf{C}$ with a wide subcategory $\mathsf{W}$ of **weak equivalences** such that $\mathsf{W}$ is the kernel of some functor out of $\mathsf{C}$, i.e. there is a category $\mathsf{D}$ and a functor $F : \mathsf{C} \to \mathsf{D}$ such that $f \in \mathsf{W}$ iff $F(f)$ is an isomorphism in $\mathsf{D}$. A morphism $f \in \mathsf{W}$ is called a **weak equivalence**.

**homotopical category**

**weak equivalence**

The reference to 'some functor $F$' in this definition might look a bit uneasy. Fortunately every subcategory of weak equivalences determines such a functor canonically, as the following shows:

**5.12.** Recall that for any set of morphisms $W$ of a category $\mathsf{C}$ there is a category $\mathsf{C}[W^{-1}]$ and a canonical functor $\gamma : \mathsf{C} \to \mathsf{C}[W^{-1}]$ **localizing at** $W$, i.e. all $f \in W$ become isomorphisms. This functor has the universal property that any functor $F : \mathsf{C} \to \mathsf{D}$ localizing at $W$ induces a unique $F_! : \mathsf{C}[W^{-1}] \to \mathsf{D}$ making the following diagram commute.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathsf{D} \\
\mathsf{C}[W^{-1}] \arrow[ur, dashed, "F_!"'] &
\end{tikzcd}$$

**5.13.** For any class of morphisms $W$ the functor $\gamma : \mathsf{C} \to \mathsf{C}[W^{-1}]$ factors any other functor inverting all morphisms in $W$. Let $\overline{W}$ denote the kernel of $\gamma$, then any functor localizing at $W$ also necessarily localizes at $\overline{W}$. In other words $\overline{W}$ is the smallest category of weak equivalences on $\mathsf{C}$ containing all the morphisms $W$. In this case we say that $W$ generates the weak equivalences $\overline{W}$. A category $\mathsf{C}$ with weak equivalences $\mathsf{W}'$ is of small generation if there is a (small) set $W$ such that $\mathsf{W}' = \overline{W}$.

**5.14.** When $\mathsf{C}$ is a homotopical category with weak equivalences $\mathsf{W}$ we write $\mathsf{Ho}(\mathsf{C})$ for $\mathsf{C}[\mathsf{W}^{-1}]$ and call it the **homotopy category** of $\mathsf{C}$. The weak equivalences are usually left implicit when speaking of a homotopical category. Just as for categories with homotopies we can equip any ordinary category with a **minimal homotopical structure**, this is equivalently the kernel of the identity functor or the weak equivalences generated by the empty set.

**homotopy category**

**5.15.** A functor $F : \mathsf{C} \to \mathsf{D}$ between homotopical categories is called **homotopical** if it sends weak equivalences in $\mathsf{C}$ to weak equivalences in $\mathsf{D}$.

**5.16.** Just as for a category with homotopies considered above the homotopy category can be used to detect when functors are homotopical. Indeed, a functor $F : \mathsf{C} \to \mathsf{D}$ is homotopical iff if factors as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "F"] \arrow[d, "\gamma"] & \mathsf{D} \arrow[d, "\gamma"] \\
\mathsf{Ho}(\mathsf{C}) \arrow[r, dashed] & \mathsf{Ho}(\mathsf{D})
\end{tikzcd}$$

**5.17.** Given a category $\mathsf{D}$ and a homotopical category $\mathsf{C}$ with a functor $F : \mathsf{D} \to \mathsf{C}$ there is a canonical choice of weak equivalences on $\mathsf{D}$ such that $F$ becomes homotopical. This defines a morphism $f \in \mathsf{D}$ to be a weak equivalence iff $F(f)$ is a weak equivalence in $\mathsf{C}$. This indeed defines a subcategory of weak equivalence, it is the kernel of $\gamma F : \mathsf{D} \to \mathsf{Ho}(\mathsf{C})$. This shows that we can induce homotopical structures (backwards) over functors.

**5.18.** The main motivation for introducing homotopical categories is the following: Let $\mathcal{T}\mathrm{op}_q$ be the category $\mathcal{T}\mathrm{op}$ with the weak equivalences from 4.19. This is a homotopical category by the functor $\pi$ from 4.20 sending a space to its (graded) homotopy groupoids. Recall that this means that a morphism $f : A \to B$ is a weak equivalence if $\pi_n(f) : \pi_n(A) \to \pi_n(B)$ is an isomorphism of groupoids for all $n \geq 0$.

**5.19.** Any category of weak equivalences $\mathsf{W}$ satisfies the **two-out-of-six property**: given three composable arrows displayed below left with their compositions, if $gf$ and $hg$ in $\mathsf{W}$ then $f, g, h$ also in $\mathsf{W}$ (hence so is $hgf$).

**two-out-of-six property**

$$\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[dr, "gf"'] \arrow[drr, dotted] & \cdot \arrow[d, "g"] \arrow[dr, "hg"] & \\
& \cdot \arrow[r, "h"] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[dr, "gf"'] & \cdot \arrow[d, "g"] \\
& \cdot
\end{tikzcd}$$

It also satisfies the two-out-of-tree property: given two composable arrows $f$ and $g$ such that $gf \in \mathsf{W}$ then $f \in \mathsf{W}$ iff $g \in \mathsf{W}$. This is an easy consequence of the two-out-of-six property.

*Proof.* We work in $\mathsf{Ho}(\mathsf{C})$ and show that $g$ has an inverse there. By assumption there are inverses $(gf)^{-1}$ of $(hg)^{-1}$ now $(hg)^{-1}h$ is left inverse of $g$ and $f(gf)^{-1}$ is a right inverse of

$f$ so then $g$ is an isomorphism. Now clearly if $gf$ and $g$ are isomorphisms in $\mathsf{Ho}(\mathsf{C})$ then so is $f$ and similarly for $h$. $\square$

**5.20.** Conversely not every wide subcategory satisfying the two-out-of-six property defines a subcategory of weak equivalences. Instead it should satisfy even more elaborate closure properties enjoyed by isomorphisms, the two-out-of-three and two-out-of-six properties can be seen as the first rungs in this ladder. In practice the two-out-of-six property represents a convenient observation to make.

**5.21.** Suppose we have a category $\mathsf{C}$ with homotopy relation $\sim$ and an homotopy invariant functor $F : \mathsf{C} \to \mathsf{D}$. Then the homotopy equivalences of $\mathsf{C}$ are included in $\mathsf{ker}(F)$, in other words $\mathsf{ker}(F)$ are weak equivalences on $\mathsf{C}$ coarser than the homotopy equivalences induced by $\sim$.

*Proof.* Consider a homotopy equivalence $f$ with homotopy inverse $g$, i.e. $fg \sim \mathsf{id}$ and $gf \sim \mathsf{id}$. Then $F(fg) = F(\mathsf{id}) = \mathsf{id}$ and $F(gf) = F(\mathsf{id}) = \mathsf{id}$ are isomorphism, so $gf$ and $fg$ are weak equivalences hence by two-out-of-six so are $f$ and $g$.

$$\begin{tikzcd}
\cdot \arrow[r, "f" description] \arrow[dr, "gf" description] \arrow[drr, dotted] & \cdot \arrow[d, "g" description] \arrow[dr, "fg" description] & \\
& \cdot \arrow[r, "f" description] & \cdot
\end{tikzcd}$$

$\square$

**5.22.** For a category $\mathsf{C}$ with homotopy relation the homotopy equivalences are exactly the maps sent to isomorphisms in $h\mathsf{C}$ we conclude that $h\mathsf{C} \cong \mathsf{Ho}(\mathsf{C})$ where the weak equivalences are the homotopy equivalences. This means that studying categories with equivalences is strictly more powerful than studying categories with homotopies. On the other hand homotopies are more intuitive and easier to reason.

**5.23.** If $\mathsf{A}$ is an ordinary category and $\mathsf{C}$ is an homotopical category then the functor category $\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{C})$ inherits a homotopical structure from $\mathsf{C}$ where an natural transformation $\theta : F \Rightarrow G$ is a weak equivalence if for all $a \in \mathsf{A}$ the component $\theta_a : F(a) \to G(a)$ is a weak equivalence.

**5.24.** Recall that in an ordinary category the limit of an functor $F : \mathsf{A} \to \mathsf{C}$ represents $\mathsf{Nat}(\Delta(-), F)$. Here $\Delta$ sends an object $C \in \mathsf{C}$ to the constant functor. Dually the colimit represents $\mathsf{Nat}(F, \Delta(-))$. We can summarize this situation by the adjoint triple

$$\begin{tikzcd}
\mathsf{C}^{\mathsf{A}} \arrow[r, bend left=40, "\mathsf{colim}"] \arrow[r, bend right=40, "\mathsf{lim}"'] \arrow[r, phantom, shift left=2.5ex, "\perp"] \arrow[r, phantom, shift right=2.5ex, "\perp"] & \mathsf{C} \arrow[l, "\Delta" description]
\end{tikzcd}$$

**5.25.** The functor $\Delta$ is homotopical however the adjoints $\mathsf{colim}$ and $\mathsf{lim}$ definitely do not have to be homotopical. To illustrate this we show an example in $\mathcal{T}\mathrm{op}_q$ involving $\mathsf{colim}$. Consider the following diagrams where the right hand diagram is obtained after replacing by weak equivalence $D^2 \to \mathbf{1}$.

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, dashed] \\
\mathbf{1} \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbf{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
S^1 \arrow[r, "i"] \arrow[d, "i"] & D^2 \arrow[d, dashed] \\
D^2 \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & S^2
\end{tikzcd}$$

Now clearly the pushouts of these diagrams as displayed are not weakly equivalent, it is actually the right hand diagram that shows the correct homotopical colimit, which we define in the next chapter. A similar example displayed below shows that the correct homotopical colimit (displayed on the right) of a diagram involving discrete topological spaces (displayed on the left) no longer needs to be discrete!

$$\begin{tikzcd}
\mathbf{1} \sqcup \mathbf{1} \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, dashed] \\
\mathbf{1} \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbf{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial I \arrow[r, "i"] \arrow[d, "i"] & I \arrow[d, dashed] \\
I \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & S^1
\end{tikzcd}$$

## 5.3 Derived functors and deformations

**right extension**

**5.26.** Given a functor $F : \mathsf{C} \to \mathsf{D}$ and a functor $i : \mathsf{C} \to \mathsf{C}'$ a **right extension** of $F$ along $i$ is a functor $\tilde{F} : \mathsf{C}' \to \mathsf{D}$ equipped with an natural transformation $\tilde{F}i \Rightarrow F$. Similarly an left extension comes with an natural transformation the other way.

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, "F"{name=F}] \arrow[dr, "i"'] & & \mathsf{D} \\
& \mathsf{C}' \arrow[ur, "\tilde{F}"'] \arrow[u, Rightarrow, shorten=2mm] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{C} \arrow[rr, "F"] \arrow[dr, "i"'] & & \mathsf{D} \\
& \mathsf{C}' \arrow[ur, "\mathsf{Ran}_i F"] \arrow[ur, bend right, "\tilde{F}"'] &
\end{tikzcd}$$

**right Kan extension**

There is an obvious notion of an universal right extension which is called the **right Kan extension** $\mathsf{Ran}_i F$ of $F$ along $i$. This is an extension such that the natural transformation of any other extension $\tilde{F}i \Rightarrow F$ correspond bijectively with natural transformations $\tilde{F} \Rightarrow \mathsf{Ran}_i F$. There is an obvious dual notion for left Kan extension, for more information see [Rie19].

**total left derived functor**

**5.27.** Consider the context of an ordinary functor between two homotopical categories now if $F$ fails to be homotopical it does not descend to a functor between the homotopy categories. But we might attempt to approximate $F$ using a left or right kan extension as follows. These approximation are called the **total left derived functor** $\mathbf{L}F$ and **total right derived functor** $\mathbf{R}F$. These are defined as follows

$$\mathbf{L}F = \mathsf{Ran}_{\gamma_\mathsf{C}} \gamma_D F, \qquad \mathbf{R}F = \mathsf{Lan}_{\gamma_\mathsf{C}} \gamma_D F$$

Note here the unfortunate switch in terminology: the total left derived functor is given by the right kan extension and vice versa.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "F"] \arrow[d, "\gamma_{\mathsf{C}}"] \arrow[dr, phantom, "\Uparrow"] & \mathsf{D} \arrow[d, "\gamma_{\mathsf{D}}"] & \mathsf{C} \arrow[r, "F"] \arrow[d, "\gamma_{\mathsf{C}}"] \arrow[dr, phantom, "\Downarrow"] & \mathsf{D} \arrow[d, "\gamma_{\mathsf{D}}"] \\
\mathsf{Ho}(\mathsf{C}) \arrow[r, "\mathbf{L}F"'] & \mathsf{Ho}(\mathsf{D}) & \mathsf{Ho}(\mathsf{C}) \arrow[r, "\mathbf{R}F"'] & \mathsf{Ho}(\mathsf{D})
\end{tikzcd}$$

**5.28.** In good situations the total derived functor lifts to an homotopical functor $\mathbb{L}F : \mathsf{C} \to \mathsf{D}$ with a natural transformation $\mathbb{L}F \Rightarrow F$ such that the induced map between the homotopy categories is the total derived functor. Such a lift is called the **left/right derived functor**.

**left/right derived functor**

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "\mathbb{L}F"'] \arrow[r, phantom, "\Uparrow"] \arrow[d, "\gamma_{\mathsf{C}}"] & \mathsf{D} \arrow[d, "\gamma_{\mathsf{D}}"] \\
\mathsf{Ho}(\mathsf{C}) \arrow[r, "\mathbf{L}F"'] & \mathsf{Ho}(\mathsf{D})
\end{tikzcd}$$

**5.29.** We can now make precise what we meant by homotopy (co)limit in 5.25. The homotopical colimit $\mathsf{hocolim}$ is a left derived functor for $\mathsf{lim}$ and the homotopical limit $\mathsf{holim}$ is the right derived functor for $\mathsf{colim}$. For more info on homotopical (co)limits and how to compute them, see [Rie19].

**5.30.** A **left deformation** on an homotopical category $\mathsf{C}$ is an endofunctor $Q$ and a natural weak equivalence $q : Q \Rightarrow \mathsf{id}$. We say that $(Q, q)$ is a left deformation for a functor $F : \mathsf{C} \to \mathsf{D}$ between homotopical categories if $FQ$ is an left derived functor with $Fq : FQ \Rightarrow F$. Dually an **right deformation** is an endofunctor $R : \mathsf{C} \to \mathsf{C}$ with natural weak equivalence $r : \mathsf{id} \Rightarrow R$.

**left deformation**

**5.31.** An **deformable adjunction** between homotopical categories is an adjunction $F \dashv G$ such that both functors are deformable. Then by [Dwy+05, 44.2] the the induced functors $\mathbf{L}F \vdash \mathbf{R}G$ between the homotopical categories are also an adjunction. In particular when this induced adjunction is an equivalence we will say that $\mathsf{C}$ and $\mathsf{D}$ are homotopically equivalent.

**deformable adjunction**

## 5.4 Model categories

**5.32.** The relation between a $\mathsf{C}$ with weak equivalences and $\mathsf{Ho}(\mathsf{C})$ is much more hairy than the relation between a category with homotopy relation and its naive homotopy category. Consider for example the category $\mathsf{C}$ displayed below left and the category $M_{\mathbb{N}}$ freely generated by the graph displayed below right.

$$\begin{tikzcd}
L \arrow[r, bend left, "s"] \arrow[r, bend right, "f"'] & R & M \arrow[loop left, "s"]
\end{tikzcd}$$

There is a functor $F$ such that $F(L) = F(R) = M$, $F(f) = \mathsf{id}_M$ and $F(s) = s$, then the kernel of this defines a homotopical structure on $\mathsf{C}$ such that $\{\mathsf{id}_L, f, \mathsf{id}_R\}$ are weak equivalences. In fact $F : \mathsf{C} \to M_{\mathbb{N}}$ is isomorphic to $\gamma : \mathsf{C} \to \mathsf{Ho}(\mathsf{C})$. Now there are maps, such as $s^n$ for $n > 1$

in $M_{\mathbb{N}}$ that have no preimage in $\mathsf{C}$. This does not happen for categories $\mathsf{D}$ with homotopy relation $\sim$: every isomorphism class of maps $[f]$ in $h\mathsf{D}$ clearly has some representative in $\mathsf{D}$.

The structure of a model category can be seen as a way to partially recover a compatible homotopy relation between some of the objects in $\mathsf{Ho}(\mathsf{C})$.

**model structure**

**5.33.** A **model structure** on a category $\mathsf{C}$ with weak equivalences $\mathsf{W}$ consists of two wide subcategories $\mathcal{F}\mathrm{ib}$ and $\mathcal{C}\mathrm{of}$ of $\mathsf{C}$ such that the pairs $(\mathcal{C}\mathrm{of} \cap \mathsf{W}, \mathcal{F}\mathrm{ib})$ and $(\mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib} \cap \mathsf{W})$ are both weak factorization systems.

**5.34.** Some terminology

- A map $p : E \to B$ in $\mathcal{F}\mathrm{ib}$ is called a **fibration**
- If $p$ is also in $\mathsf{W}$ then it is a **trivial/acyclic fibration**
- A map $i : A \to X$ in $\mathcal{C}\mathrm{of}$ is called a **cofibration**
- If $i$ is also in $\mathsf{W}$ then it is a **trivial/acyclic cofibration**

The model category axioms imply that, for every square displayed below left such that $i$ is a cofibration and $p$ is a fibration, that if either $i$ or $p$ is in $\mathsf{W}$ there exists a diagonal filler.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\tilde{i}"] \arrow[d, "i"] \arrow[dr, "f"] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "\tilde{p}"] & B \end{tikzcd}$$

Moreover every morphism $f : A \to B$ in $\mathsf{C}$ displayed above right factors in two ways such that: $\tilde{i}$ is an trivial cofibration; $p$ is a fibration; $i$ is a cofibration; $\tilde{p}$ is a trivial fibration.

**5.35.** It should be noted that model structures are not unique for a given homotopical category, nor do they have to exist.

**model category**

**5.36.** A **model category** is a complete and cocomplete category $\mathsf{C}$ together with a model structure on $\mathsf{C}$.

**5.37.** The category with homotopies $\mathcal{T}\mathrm{op}_h$ is a model category where $\mathcal{F}\mathrm{ib}$ are the Hurewicz fibrations and $\mathcal{C}\mathrm{of}$ are the (closed) Hurewicz cofibrations. When we talk about $\mathcal{T}\mathrm{op}_h$ we will always use this model structure which is called the **Hurewicz model structure** on $\mathcal{T}\mathrm{op}$.

**Hurewicz model structure**

The homotopical category $\mathcal{T}\mathrm{op}_q$ is a model category where $\mathcal{F}\mathrm{ib}$ are the Serre fibrations and $\mathcal{C}\mathrm{of}$ are the Serre cofibrations. When we talk about $\mathcal{T}\mathrm{op}_q$ we will always use this model structure which is called the **Quillen model structure** on $\mathcal{T}\mathrm{op}$.

**Quillen model structure**

**5.38.** If we examine the situation in $\mathcal{T}\mathrm{op}_h$ we see that homotopies between $f, g : A \to B$ are represented by a map $H : I \times A \to B$ such that $H(0, -) = f$ and $H(1, -) = g$. In other words we have a commutative diagram (where $\nabla$ is the fold map $[\mathsf{id}, \mathsf{id}]$).

$$\begin{tikzcd}
A+A \arrow[r, "f+g"] \arrow[d, "\nabla"'] \arrow[dr, "\partial_0+\partial_1"] & B \\
A & I\times A \arrow[l, "p"] \arrow[u, "H"']
\end{tikzcd}$$

What is to notice here is that $p$ is a homotopy equivalence and $\partial_0 + \partial_1$ is a cofibration (closed embedding) with respect to our model structure. This suggests that $I \times A$ is a kind of nice replacement (equivalent by $p$) of $A + A$ such that we can represent the homotopy between $f$ and $g$.

Abstracting the details away we get the following

**5.39.** A **cylinder** $\mathsf{Cyl}(A)$ for $A$ is a factorization of the fold map $\nabla$ as a cofibration followed by a weak equivalence $A + A \xrightarrow{i} \mathsf{Cyl}(A) \xrightarrow{\sim} A$. Dually a **path object** $\mathsf{Path}(B)$ for $B$ is factorization of the diagonal $\Delta$ as a weak equivalence followed by a fibration $B \xrightarrow{\sim} \mathsf{Path}(B) \xrightarrow{p} B \times B$. The model theory axioms always ensure that cylinders and path objects exist, we then even have that the weak equivalences are trivial fibrations/cofibrations.

**cylinder**

**path object**

An left homotopy between $f, g : A \to B$ in a model category $\mathsf{C}$ is a factorization through the cylinder object of $A$ as displayed below left. Dually a right homotopy between $f, g : A \to B$ is a factorization through the path object of $B$ displayed below right.

$$\begin{tikzcd}
A+A \arrow[r, "f+g"] \arrow[d, "i"'] & B & A \arrow[r, "f\times g"] \arrow[dr, "H"'] & B\times B \\
\mathsf{Cyl}(A) \arrow[ur, "H"] & & & \mathsf{Path}(A) \arrow[u, "p"']
\end{tikzcd}$$

Warning: left homotopy or right homotopy is not necessarily an equivalence relation on $\mathsf{Hom}(A, B)$

**5.40.** An object $A$ is **cofibrant** if the unique map from the initial object $0 \to A$ is cofibration. An object $A$ is called **fibrant** if the unique map to the terminal object $A \to 1$ is a fibration. An object is called **bifibrant** if it is both fibrant and cofibrant.

**fibrant, cofibrant and bifibrant objects**

$$\begin{tikzcd}
A \arrow[r, "\sim"] \arrow[dr, "!"] & R(A) \arrow[d, two heads] & \mathbf{0} \arrow[r, tail] \arrow[dr, "!"] & Q(A) \arrow[d, "\sim"] \\
& \mathbf{1} & & A
\end{tikzcd}$$

**5.41.** The model category axioms ensure that every object $A$ in a model category has an weakly equivalent fibrant object $R(A)$ and a weakly equivalent cofibrant $Q(A)$ as displayed above. Such objects are called **fibrant/cofibrant replacements** for $A$. Chain these operations even yields an bifibrant object $\overline{A}$, as can be easily checked.

**fibrant/cofibrant replacements**

**5.42.** If the factorization systems are functorial we even obtain **fibrant/cofibrant replacement functors** $R, Q : \mathsf{C} \to \mathsf{C}$ equipped with natural weak equivalences $\mathsf{id} \Rightarrow Q$ and $R \Rightarrow \mathsf{id}$. The bifrant objects $RQ(A)$ and $QR(A)$ are not necessarily the same, although they are of course equivalent.

**fibrant/cofibrant replacement functors**

**5.43.** When $A$ is cofibrant then left homotopy is an equivalence relation on $\mathsf{Hom}(A, B)$ for all $B$. Dually, right homotopy is an equivalence relation when $B$ is a fibrant. By a theorem of Quillen see [Qui67] left and right homotopy coincide which we will write as $\sim$ when $A$ is cofibrant and $B$ is fibrant. Quillen also showed that in this situation

$$\mathsf{Hom}_{\mathsf{Ho}(\mathsf{C})}(A, B) = \mathsf{Hom}(A, B)/\sim$$

**5.44.** Since weak equivalences are isomorphisms in the homotopy category of $\mathsf{C}$ and we actually get that

$$\mathsf{Hom}_{\mathsf{Ho}(\mathsf{C})}(X, Y) = \mathsf{Hom}(QX, RY)/\sim = \mathsf{Hom}(RQX, QRY)/\sim$$

If we write $\overline{\mathsf{C}}$ for the full subcategory of bifibrant objects then the above shows that $\mathsf{Ho}(\mathsf{C}) = h\overline{\mathsf{C}}$ where we equip $\overline{\mathsf{C}}$ with the homotopy relation $\sim$.

**5.45.** In $\mathcal{T}\mathrm{op}_q$ the bifibrant objects are given by the retracts of CW-complexes, moreover the homotopy relation produced by cylinders and path objects on bifibrant objects coincides with the homotopy relation of $\mathcal{T}\mathrm{op}_h$. This means that the homotopy category of $\mathcal{T}\mathrm{op}_q$ is equivalent to the full subcategory of retracts of CW-complexes with the naive homotopy relation.

## 5.5 Model structure on $\mathcal{C}\mathrm{at}$

**walking isomorphism**

In the category of categories $\mathcal{C}\mathrm{at}$ is a category $\mathsf{I}$ called the **walking isomorphism** category containing two objects 0 and 1 and an isomorphism between them. It is named so because the functors $\mathsf{I} \to \mathsf{C}$ correspond bijectively with the isomorphisms of $\mathsf{C}$. This functor serves as an interval object and generates a homotopy relation on $\mathcal{C}\mathrm{at}$ in the following sense. Two parallel functors $F, G : \mathsf{C} \to \mathsf{D}$ are homotopic if there is a functor $H : I \times \mathsf{C} \to \mathsf{D}$ such that $H(0, -) = F(-)$ and $H(1, -) = G(-)$, such a functor is precisely an natural isomorphism between $F$ and $G$. A natural isomorphism can equivalently be presented by a map $\mathsf{C} \to \mathcal{F}\mathrm{un}(\mathsf{I}, \mathsf{D})$ and this makes it clear that this is indeed a homotopy relation. The homotopy equivalences are then precisely the categorical equivalences.

**isofibrations**

The homotopical category $\mathcal{C}\mathrm{at}$ can be equipped with a model structure in precisely one way[1]. This model structure is cofibrantly generated as follows. The generating trivial cofibrations is the endpoint inclusion $i : \{0\} \hookrightarrow I$, a map having lifts against this inclusion are the **isofibrations**. More explicitly, an functor $p : \mathsf{E} \to \mathsf{B}$ is an isofibration if for each $e \in E$ and isomorphism $f : p(e) \cong b$ there is an $e' \in E$ and $f' : e \cong e'$ such tat $p(f') = f$. This is displayed in the lifting square below left.

$$\begin{tikzcd}
\{1\} \arrow[r, "e"] \arrow[d, "i"] & E \arrow[d, "p"] \\
\mathsf{I} \arrow[r, "f"'] \arrow[ur, dashed, "f'"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{P} \arrow[d, "i"] \\
\mathbf{2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[d, dotted] & \cdot \arrow[d, dotted] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

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[1] https://sbseminar.wordpress.com/2012/11/16/the-canonical-model-structure-on-cat/

Recall that $\mathbf{0}$ is the category with no arrows, $\mathbf{1}$ is the category with single object and identity morphism and $\mathbf{2}$ is the category with two arrows and a single non identity arrow between them. There is a (unique) functor $! : \mathbf{0} \to \mathbf{1}$, a map has the right lifting property against ! iff it is surjective on objects. Consider the functor $\partial : \mathbf{1} + \mathbf{1} \to \mathbf{2}$ which includes into the endpoints of $\mathbf{2}$, another functor has the right lifting property against $\partial$ iff it is full. Finally consider the category $\mathsf{P} = \mathbf{2} +^{\partial}_{\partial} \mathbf{2}$ the category with two parallel arrows and the functor $s : P \to \mathbf{2}$ identifying the parallel arrows depicted above right. A functor has the right lifting property against $s$ iff it is faithful. These three maps together form the generating cofibrations, this means that a trivial fibration in $\mathcal{C}\mathrm{at}$ are precisely the full and faithful maps which are surjective on objects.

# Chapter 6

# Simplicial sets

**6.1.** Let the **simplex category** $\mathbf{\Delta}$ be (the skeleton of) the full subcategory of $\mathcal{C}\mathrm{at}$ spanned by the non empty finite linear orders. Concretely the category $\mathbf{\Delta}$ consists of linear orders $[n] = \{0 < ... < n\}$ for each $n > 0$ and all order preserving maps between them. There are some important variations on the simplex category

**simplex category**

- Let the **augmented simplex category** $\mathbf{\Delta}_+$ be the full subcategory of $\mathcal{C}\mathrm{at}$ spanned by all finite linear orders. This is just $\mathbf{\Delta}$ with an additional initial object $[-1] = \varnothing$ the empty linear order.
- For any $n > 1$ let the *n***-truncated simplex category** $\mathbf{\Delta}_{\leq n}$ be the full subcategory of $\mathbf{\Delta}$ spanned by the objects $[i]$ with $i \leq n$.

**augmented simplex category**

***n*-truncated simplex category**

The (augmented) simplex category inherits the orthogonal (epi, mono) factorization system from sets yielding two full subcategories $(\mathbf{\Delta}_+)_{\mathsf{epi}}$ and $(\mathbf{\Delta}_+)_{\mathsf{mono}}$. These two subcategories, and hence $\mathbf{\Delta}_+$ it self are generated by the maps

- The **face maps** $\left\{\delta^i_n : [n-1] \to [n] \mid 0 \leq i \leq n\right\}$ where $\delta^i_n$ is injective and leaves $i$ out of its image.
- The **degeneracy maps** $\left\{\sigma^i_n : [n+1] \to [n] \mid 0 \leq i \leq n\right\}$ where $\sigma^i_n$ is surjective and repeats the $i$ in its domain.

Similar remarks hold for $\mathbf{\Delta}$ itself and the $n$-truncated simplex categories. Drawing only these generating maps the augmented simplex category is pictured as

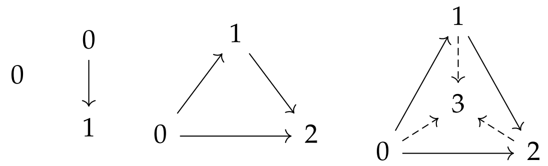
$$\begin{tikzcd}
{[-1]} \arrow[r] & {[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]} \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & {[2] \cdots} \arrow[l, shift left=2] \arrow[l, shift right=2]
\end{tikzcd}$$

An diagram $X_\bullet : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ with shape $\mathbf{\Delta}$ is called a **simplicial object** in $\mathsf{C}$. We then write $X_n$ for $X_\bullet([n])$. A diagram with shape $\mathbf{\Delta}$ is called a **cosimplicial object**. Similarly for **augmented (co)simplicial object**, *n***-truncated (co)simplicial object**, etc.

The presheaf category $\mathcal{P}\mathrm{sh}(\mathbf{\Delta})$ of functors $\mathbf{\Delta}^{\mathsf{op}} \to \mathcal{S}\mathrm{et}$ is called the **category of simplicial sets** and is also written $\mathrm{s}\mathcal{S}\mathrm{et}$. The most important way to think about the category of simplicial

**category of simplicial sets**

sets is that it is the cocompletion of $\mathbf{\Delta}$, i.e. every simplicial set $X \in \mathcal{sSet}$ is obtained by gluing together representable simplices $\mathbf{y}[n]$ which are also written as $\Delta[n]$. The representables $\Delta[n]$ should be thought of as abstract oriented $n$-simplices as displayed below where the faces/volumes should be imagined as filled in. The face and degeneracy maps induce injections and surjections between these standard simplices.



For a subcategory $i : \mathbf{\Delta}_{\leq n} \hookrightarrow \mathbf{\Delta}$ there is an map $\mathbf{tr}_n : \mathcal{sSet} \to \mathcal{P}\mathrm{sh}(\mathbf{\Delta}_{\leq n})$ given by precomposition called the truncation map. This map has a left adjoint $\mathbf{sk}_n : \mathcal{P}\mathrm{sh}(\mathbf{\Delta}_{\leq \mathsf{n}}) \to \mathcal{sSet}$ obtained by the following construction. We can consider the functor $\mathbf{y}i : \mathbf{\Delta}_{\leq n} \to \mathcal{sSet}$ which is a functor into a cocomplete category $\mathcal{sSet}$, by extensions by colimits we then get the adjoint pair

$$\begin{tikzcd}
\mathcal{P}\mathrm{sh}(\mathbf{\Delta}_{\leq \mathsf{n}}) \arrow[r, bend left, "\mathbf{sk}_n"] \arrow[r, phantom, "\perp"] & \mathcal{sSet} \arrow[l, bend left, "\mathbf{tr}_n"]
\end{tikzcd}$$

This adjunction induces a functors which we will also call $\mathbf{sk}_n : \mathcal{sSet} \to \mathcal{sSet}$. This functor forgets all the higher simplices. An simplicial set in the essential image of $\mathbf{sk}_n$ will be called an $n$-truncated simplicial set.

The category of simplicial sets is an important middle ground between topological spaces and categories as the following shows.

## Spaces and simplicial sets

**6.2.** Define a functor $\mathbf{\Delta} \to \mathcal{T}\mathrm{op}$ which sends $[n] \mapsto \{x \in I^n \mid 0 \leq x_1 \leq ... \leq x_n \leq 1\}$ **the standard $n$-simplex** in $\mathcal{T}\mathrm{op}$ where the face and degeneracy maps are the obvious ones. Since $\mathcal{T}\mathrm{op}$ is cocomplete we obtain an adjoint pair using extension by colimits:

$$\begin{tikzcd}
\mathcal{sSet} \arrow[r, bend left, "|-|_{\mathcal{T}\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathcal{T}\mathrm{op} \arrow[l, bend left, "\mathsf{Sing}"]
\end{tikzcd}$$

The left adjoint $|-|_{\mathcal{T}\mathrm{op}} : \mathcal{sSet} \to \mathcal{T}\mathrm{op}$ is called the **geometric realization** functor and is obtained by: 1) decomposing a simplicial set as a colimit of representables; 2) taking a standard simplex for every representable; 3) gluing the standard simplices together according to the colimit presentation. The right adjoint $\mathsf{Sing} : \mathcal{T}\mathrm{op} \to \mathcal{P}\mathrm{sh}(\mathbf{\Delta})$ sends a topological space to it **singular simplicial complex** which is obtained by probing a space with the standard simplices. The image of the geometric realization functor are CW complexes, as should be clear. Then standard homotopy theory shows that the counit $\epsilon_X : |\mathsf{Sing}X| \to X$ is an weak homotopy equivalence.

**6.3.** The category $\mathrm{s}\mathcal{S}\mathrm{et}$ can be endowed with a homotopical structure where we let the weak equivalences be those induced by the geometric realization functor $|-|_{\mathcal{T}\mathrm{op}}$ (which is then homotopical by definition). This homotopical structure is called the **Quillen homotopical structure on simplicial sets** and we will write it as $\mathrm{s}\mathcal{S}\mathrm{et}_q$.

**6.4.** Since the counit's are weak homotopy equivalences the composite $|\mathsf{Sing} \bullet|$ is also homotopical, indeed suppose $f$ is a weak equivalence then $|\mathsf{Sing} f|$ is a weak equivalence by the two-out-of-three property (displayed below left). Then $\mathsf{Sing} f$ is already a weak equivalence since otherwise $|\mathsf{Sing} f|$ would not be.

$$\begin{tikzcd}
{|\mathsf{Sing}X|} \arrow[r, "|\mathsf{Sing}f|"] \arrow[d, "\epsilon_X"'] & {|\mathsf{Sing}Y|} \arrow[d, "\epsilon_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
{|Y|} \arrow[r, "|\eta_Y|"] \arrow[dr, "\mathsf{id}"'] & {|\mathsf{Sing}|Y||} \arrow[d, "\epsilon_{|Y|}"] \\
 & {|Y|}
\end{tikzcd}$$

Now consider for any $Y$ the unit map $\eta_Y : Y \to \mathsf{Sing}|Y|$, by the induced homotopical structure this map is a weak equivalence if $|\eta_Y| : |Y| \to |\mathsf{Sing}|Y||$ is. The triangle law for adjunctions gives the top right triangle and since $\mathsf{id}$ and the counit are weak equivalences so is $|\eta_Y|$ by two-out-of-three.

**6.5.** This shows that the realization-singular adjunction between topological spaces and simplicial sets is homotopical and moreover induces an equivalence on the level of homotopy categories.

$$\begin{tikzcd}
\mathrm{s}\mathcal{S}\mathrm{et}_q \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] \arrow[d, "\gamma"] & \mathcal{T}\mathrm{op}_q \arrow[l, bend left, "\mathsf{Sing}"] \arrow[d, "\gamma"] \\
\mathsf{Ho}(\mathrm{s}\mathcal{S}\mathrm{et}_q) \arrow[r, bend left, dashed] \arrow[r, phantom, "\perp"] & \mathsf{Ho}(\mathcal{T}\mathrm{op}_q) \arrow[l, bend left, dashed]
\end{tikzcd}$$

**6.6.** There is a model structure on $\mathrm{s}\mathcal{S}\mathrm{et}$ compatible with the weak equivalences induced by $|-|_{\mathcal{T}\mathrm{op}}$, we will also write $\mathrm{s}\mathcal{S}\mathrm{et}_q$ for this model structure. This model structure is cofibrantly generated where

- The generating cofibrations are given by the boundary inclusions $i : \partial\Delta_n \hookrightarrow \Delta_n$ for all $0 \leq n$. Here $\partial\Delta_n$ is the subobject of $\Delta_n$ induced by all the face maps into $\delta_n^\bullet : [n-1] \to [n]$.
- The generating trivial cofibrations are given by the horn inclusions $\Lambda_n^i \hookrightarrow \Delta_n$ for $0 \leq i \leq n$. Here a horn $\Lambda_n^i$ is the union of the subobjects of $\Delta_n$ induced by all but the $i$th face maps $\delta_n^\bullet : [n-1] \to [n]$.

**6.7.** The image of the generating trivial cofibrations are precisely the boundary inclusions of spheres-into-discs (up to isomorphism) as displayed below right. Suppose that $f$ is a Serre fibration, i.e. $f$ has lifts against all spheres-into-discs. Then any lifting problem of a generating cofibration in $\mathrm{s}\mathcal{S}\mathrm{et}$ into $\mathsf{Sing}(f)$ transposes to a lifting problem in $\mathcal{T}\mathrm{op}$ against a sphere-into-disc inclusion for which we can transport the lift back along the adjunction.

This shows that if $f$ is a fibration so will $\mathsf{Sing}(f)$ be a fibration.

$$\begin{tikzcd}
{|\partial\Delta_n|} \arrow[r, "\sim"] \arrow[d, hook] & S^{n-1} \arrow[d] & {|\partial\Lambda_n^i|} \arrow[r, "\sim"] \arrow[d, hook] & D^{n-1} \arrow[d] \\
{|\Delta_n|} \arrow[r, "\sim"] & D^n & {|\Delta_n|} \arrow[r, "\sim"] & D^{n-1} \times I
\end{tikzcd}$$

The generating cofibrations have a relation to the cylinder-of-disc inclusions as displayed above right. Then a similar argument shows that if $f$ is a trivial fibration then $\mathsf{Sing}(f)$ is also a trivial fibration.

This already shows that $|-| \vdash \mathsf{Sing}$ is a Quillen adjunction, indeed it is a Quillen equivalence since the adjunction induces an equivalence on homotopy categories.

## Categories and simplicial sets

**6.8.** Let $\mathbf{\Delta} \hookrightarrow \mathcal{C}\mathrm{at}$ be the inclusion into categories (recall that $\mathbf{\Delta}$ is just a subcategory of $\mathcal{C}\mathrm{at}$). Since $\mathcal{C}\mathrm{at}$ is cocomplete we obtain an **categorical realization** functor $|-|_{\mathcal{C}\mathrm{at}} : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$ sending an simplicial set $S$ to a category $|S|$. The left adjoint of this functor is called the **nerve** $N : \mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$. **nerve**

$$\begin{tikzcd}
\mathrm{s}\mathcal{S}\mathrm{et} \arrow[r, bend left, "|-|_{\mathcal{C}\mathrm{at}}"] \arrow[r, phantom, "\perp"] & \mathcal{C}\mathrm{at} \arrow[l, bend left, "N"]
\end{tikzcd}$$

**6.9.** The nerve of category $N(\mathsf{C})$ is a simplicial set where $[n] \mapsto \mathsf{Hom}(\Delta[n], \mathsf{C})$ this means that; $N(\mathsf{C})_0$ are the objects of $\mathsf{C}$; $N(\mathsf{C})_1$ are the morphisms of $\mathsf{C}$; $N(\mathsf{C})_2$ are composable pairs of $\mathsf{C}$; and in general $N(\mathsf{C})_n$ are composable $n$-tuples of arrows in $\mathsf{C}$.

**6.10.** The realization functor $|-|_{\mathcal{C}\mathrm{at}}$ can profitably be understood as freely generating a category from a simplicial set $S$ where the points $S_0$ are the objects of $|S|$, the 1-simplices $S_1$ generate the morphisms of $|S|$, and the 2-simplices $S_2$ quotient the resulting category by forcing the sides of $S_2$ to become equal. In the case where $S$ only has degenerate $n$-simplices for $n > 1$ can nicely be compared to generating a category from a reflexive graph as follows.

The category of reflective graphs $\mathsf{rGraph}$ is just the category $\mathcal{P}\mathrm{sh}(\mathbf{\Delta}_{\leq 1})$, in particular a reflective graph $X \in \mathsf{rGraph}$ is just a pair of sets $X_0$ and $X_1$ with reflexivity map $r : X_0 \to X_1$ and source and target maps $s, t : X_1 \to X_0$. The adjunctions between $\mathsf{rGraph}$, $\mathcal{C}\mathrm{at}$ and $\mathrm{s}\mathcal{S}\mathrm{et}$ assemble into a square as displayed below.

$$\begin{tikzcd}
\mathsf{rGraph} \arrow[r, equal] \arrow[d, shift right, "F"'] \arrow[d, phantom, "\dashv"] & \mathcal{P}\mathrm{sh}(\mathbf{\Delta}_{\leq 1}) \arrow[d, shift right, "\mathbf{sk}_1"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{C}\mathrm{at} \arrow[u, shift right, "U"'] \arrow[r, shift right, "N"'] \arrow[r, phantom, "\perp"] & \mathrm{s}\mathcal{S}\mathrm{et} \arrow[u, shift right, "\mathbf{tr}_1"'] \arrow[l, shift right, "|-|_{\mathcal{C}\mathrm{at}}"']
\end{tikzcd}$$

In fact the subsquares of left (right) adjoints commute. By uniqueness of ajoints it is enough to show that one of the squares commute. It should be clear that $\mathbf{tr}_1 \circ N$ yields precisely the underlying reflexive graph of a category. From this we conclude that if a simplicial set $S$ is 1-truncated it's realization in $\mathcal{C}\mathrm{at}$ corresponds to the free category generated by it's 1-simplices.

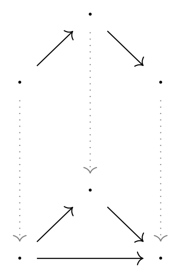
For a general simplicial set $S$ the geometric realization $|S|_{\mathcal{C}\mathrm{at}}$ can be constructed by first building the free category $F(\mathbf{tr}_1(S))$ on the 1-skeleton of $S$. Then there is homotopy relation on $F(\mathbf{tr}_1(S))$ generated by the 2-simplices of $S$ as follows. For each $\sigma \in S_2$ with faces $f, g, h$ which we can consider in $F(\mathbf{tr}_1(S))$ we require $gf \sim h$. Then $|S|$ is the quotient of $F(\mathbf{tr}_1(S))$ by this equivalence relation.

**6.11.** The counit $\epsilon_{\mathsf{C}} : |N(\mathsf{C})|_{\mathcal{C}\mathrm{at}} \to \mathsf{C}$ is the identity on objects map that sends an equivalence class of composable arrows to their composite in $\mathsf{C}$ is an isomorphism of categories. The unit of adjunction is not an equivalence in any reasonable sense, indeed it destroys all the non degenerate $n$-simplices for $n > 1$. We improve this adjunction in 7.36.

The essential image of $N$ consists of the simplicial sets can also be characterized by a lifting property. Recall that a horn inclusion's were the collection of maps $\Lambda^i_n \to \Delta_n$ for all $0 \leq i \leq n$ and $n$. For a nerve of a category $\mathsf{C}$ the unique map $N\mathsf{C} \to \Delta[0]$ has the unique right lifting property against inner horns which we now define.

**6.12.** The **inner horns** are maps $\Lambda^i_n \to \Delta_n$ for all $0 < i < n$ (notice the inequalities). **inner horns**

**6.13.** There is only one inner horn for $n = 2$ which is represented below left, and a lifting problem against this horn is displayed below right.



$$\begin{tikzcd}
\Lambda^1[2] \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
\Delta[2] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

When $B = \mathbf{1}$ and $E = N(\mathsf{C})$ the map $\Lambda^1[2] \to N(\mathsf{C})$ transposes to $\overline{f} : |\Lambda^1[2]| \to \mathsf{C}$, since $| - |$ is a left adjoint it preserves colimits so $|\Lambda^1[2]| \cong \mathbf{2} + \mathbf{2}$ and $\overline{f}$ simply picks out a pair of composable arrows. Since $\mathsf{C}$ is a category this pair has a composite which gives the desired extension of $f$ producing the diagonal filler. Since composites in a category are unique this lift is also unique.

# Chapter 7

# Topological, simplicial and quasi categories

There is an alternative approach for doing homotopical mathematics which will eventually lead to higher category theory. This is a refinement of the idea of categories with homotopies and builds on the homotopical categories $\mathcal{T}\mathrm{op}_q$ and $\mathrm{s}\mathcal{S}\mathrm{et}$.

Recall that a category with homotopies consists of an ordinary category $\mathsf{C}$ with a homotopy relation on each of the hom set $\mathsf{Hom}(A, B)$, compatible with composition. In other words each hom is actually a setoid i.e. a set with an equivalence relation, and the categorical operations such as composition are setoid morphisms. A category taking for which the hom sets naturally have the structure of objects of another category $\mathcal{E}$ is called an $\mathcal{E}$-enriched category. The idea is to now to consider categories enriched in the homotopical categories $\mathrm{s}\mathcal{S}\mathrm{et}_q$ or $\mathcal{T}\mathrm{op}_q$. This means that for each pair of objects $X, Y$ we get a simplicial set or topological space of morphisms from $A$ to $B$. Two morphisms in such a hom space can then be declared homotopical if they are in the same path component.

We will begin making the above sketch precise by introducing the notion of an enriched category, an variation on category theory where we replace $\mathcal{S}\mathrm{et}$ with some other category $\mathcal{E}$.

## 7.1 Enriched categories

For now it is good to read the following definition with $\mathcal{E} = \mathcal{S}\mathrm{et}$. For this case the reader can verify it is equivalent to the ordinary definition of a (locally small) category. We will show later how to vary $\mathcal{E}$ to obtain novel enriched categories.

**enriched category**

**7.1.** A **category enriched over** $\mathcal{E}$, or $\mathcal{E}$-category, $\mathsf{C}$ consists of

- A collection $X, Y, Z \in \mathsf{C}$ of objects, written $\mathcal{O}\mathrm{bj}(\mathsf{C})$.

- For each pair of objects $X, Y$ an **hom-object** $\underline{\mathsf{Hom}}(A, B) \in \mathcal{E}$
- For each object $X$ a morphism $1_X : * \to \underline{\mathsf{Hom}}(A, B)$ in $\mathcal{E}$
- For each triple of objects $X, Y, Z \in \mathsf{C}$ there is a morphism $\circ : \underline{\mathsf{Hom}}(Y, Z) \times \underline{\mathsf{Hom}}(X, Y) \to \underline{\mathsf{Hom}}(X, Z)$.

**hom-object**

Subject to the categorified associativity and identity laws, i.e. the following squares commute in $\mathcal{E}$

$$\begin{tikzcd}
\underline{\mathsf{Hom}}(X,Y) \times \mathbf{1} \arrow[d, "\mathrm{id}\times 1_X"] & \underline{\mathsf{Hom}}(X,Y) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, equal] & \mathbf{1} \times \underline{\mathsf{Hom}}(X,Y) \arrow[d, "1_Y\times\mathrm{id}"] \\
\underline{\mathsf{Hom}}(X,Y) \times \underline{\mathsf{Hom}}(X,X) \arrow[r, "\circ"] & \underline{\mathsf{Hom}}(X,Y) & \underline{\mathsf{Hom}}(Y,Y) \times \underline{\mathsf{Hom}}(X,Y) \arrow[l, "\circ"']
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathsf{Hom}}(Z,W) \times \underline{\mathsf{Hom}}(Y,Z) \times \underline{\mathsf{Hom}}(X,Y) \arrow[r, "\mathrm{id}\times\circ"] \arrow[d, "\circ\times\mathrm{id}"] & \underline{\mathsf{Hom}}(Z,W) \times \underline{\mathsf{Hom}}(X,Z) \arrow[d, "\circ"] \\
\underline{\mathsf{Hom}}(Y,W) \times \underline{\mathsf{Hom}}(X,Y) \arrow[r, "\circ"] & \underline{\mathsf{Hom}}(X,W)
\end{tikzcd}$$

**7.2.** The category $\mathcal{E}$ in the above definition is called the **base of enrichment** and for the above definition to make sense require that it is a symmetric monoidal category. A **symmetric monoidal category** $\mathcal{E}$ is a category with an functor $\times : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ and an object $\mathbf{1} \in \mathcal{E}$ such that there are specific isomorphisms

**symmetric monoidal category**

- $\mathbf{1} \times X \cong X \cong X \times \mathbf{1}$ for all $X \in \mathcal{E}$, witnessing that $\mathbf{1}$ is a (weak) unit of $\times$.
- $A \times (B \times C) \cong (A \times B) \times C$ for all $A, B, C \in \mathcal{E}$, witnessing that $\times$ is (weakly) associative.
- $A \times B \cong B \times A$ for all $A, B \in \mathcal{E}$, witnessing that $\times$ is (weakly) symmetric.

**7.3.** Any category with finite products is symmetric monoidal with the binary product and the terminal object. The added structure from the finite limits is that we get a diagonal map $\Delta_X : X \to X \times X$ and an augmentation map $e_X : X \to 1$ satisfying the coherence law displayed.

$$\begin{tikzcd}
& X \arrow[d, "\Delta_X"] & \\
\mathbf{1} \otimes X \arrow[ur, bend left, "\cong"] & X \times X \arrow[l, "\mathrm{id}\times e_X"'] \arrow[r, "e_X\times\mathrm{id}"] & X \otimes \mathbf{1} \arrow[ul, bend right, "\cong"']
\end{tikzcd}$$

**7.4.** Given a symmetric monoidal structure $(\mathsf{C}, \otimes, \mathbf{1})$, if the functor $- \otimes X : \mathsf{C} \to \mathsf{C}$ has a right adjoint for all $X$ we say that the monoidal structure of $\mathsf{C}$ is **closed**. In this case the right adjoint is written $[X, -]$ such that $\mathsf{Hom}(X \times Y, Z) \cong \mathsf{Hom}(X, [X, Y])$ natural in $X, Y, Z \in \mathsf{C}$.

**closed monoidal**

**7.5.** If a category $\mathsf{C}$ has a closed symmetric monoidal structure it is enriched over itself with $\underline{\mathsf{Hom}}(X, Y) := [X, Y]$ (see [Rie19, Chapter 3]). For example the enriched composition is defined to be the transposition of the map displayed below

$$[Y, Z] \otimes [X, Y] \otimes X \xrightarrow{\mathrm{id}\otimes\epsilon} [Y, Z] \otimes Y \xrightarrow{\epsilon} Z$$

The categories $\mathcal{T}\mathrm{op}$ and $\mathrm{s}\mathcal{S}\mathrm{et}$ are both cartesian closed and are thus self enriched. The convenient category of spaces $\mathcal{T}\mathrm{op}$ was chosen to be cartesian closed mostly for this reason.

**7.6.** Consider an $\mathcal{T}\mathrm{op}$-enriched category $\mathsf{C}$ and two parallel morphisms $f, g : X \to Y$ in $\mathsf{C}$. These two arrows are represented by two points $\ulcorner f \urcorner, \ulcorner g \urcorner : \mathbf{1} \to \underline{\mathsf{Hom}}(Y, X)$ in $\mathcal{T}\mathrm{op}$. This allows us to define an **homotopy** between $f$ and $g$ to be an path $H : I \to \underline{\mathsf{Hom}}(Y, X)$ such that $H(0) = \ulcorner f \urcorner$ and $H(1) = \ulcorner g \urcorner$. The same definition works in $\mathrm{s}\mathcal{S}\mathrm{et}$ where we use $\Delta[1]$ as interval. The diagram below left depicts this situation for both $\mathrm{s}\mathcal{S}\mathrm{et}$ and $\mathcal{T}\mathrm{op}$ enrichment.

**homotopy**

$$\begin{tikzcd}
\mathbf{1} \arrow[d, "i_0"'] \arrow[dr, bend left, "\ulcorner f \urcorner"] & & X \arrow[d, "i_0"'] \arrow[dr, bend left, "f"] & & & & Y \\
I \arrow[r, "H"] & \underline{\mathsf{Hom}}(X,Y) & I \cdot X \arrow[r, "H"] & Y & X \arrow[urr, bend left, "f"] \arrow[rr, "H"] \arrow[drr, bend right, "g"'] & & Y^I \arrow[u, "p_0"'] \arrow[d, "p_1"] \\
\mathbf{1} \arrow[u, "i_1"] \arrow[ur, bend right, "\ulcorner g \urcorner"'] & & X \arrow[u, "i_1"] \arrow[ur, bend right, "g"'] & & & & Y
\end{tikzcd}$$

**7.7.** In the case above where $\mathsf{C} = \mathcal{T}\mathrm{op}$ (or $\mathsf{C} = \mathrm{s}\mathcal{S}\mathrm{et}$) the object on the right is just the internal Hom space $[X, Y]$, by the Hom-adjunction we get the 'left homotopy' diagram in the middle. Transposing to the other variable in the product produces the 'right homotopy' diagram on the right. Thus we see that this notion of homotopy in $\mathcal{T}\mathrm{op}$ agrees with the Hurewicz homotopical structure $\mathcal{T}\mathrm{op}_h$.

**7.8.** The nice thing about the approach above is that we immediately get higher homotopies (as homotopies between homotopies) for free. We want to find a way to internally represent left/right homotopies for a general $\mathcal{E}$-enriched category $\mathsf{C}$, not just when is self enriched. For this we require a way to 'tensor' and 'cotensor' an object $X \in \mathsf{C}$ with an object $E \in \mathcal{E}$.

**7.9.** Suppose that $\mathcal{E}$ is an closed monoidal category (such that it is self enriched) then an $\mathcal{E}$-enriched category $\mathsf{C}$ is

- **tensored** or **copowered** if there is an object $E \cdot X \in \mathcal{E}$ such that

**tensored**

$$\underline{\mathsf{Hom}}_{\mathsf{C}}(E \cdot X, Y) \cong \underline{\mathsf{Hom}}_{\mathcal{E}}(E, \underline{\mathsf{Hom}}_{\mathsf{C}}(X, Y))$$

- **cotensored** or **powered** if there is an object $Y^E$ such that

**cotensored**

$$\underline{\mathsf{Hom}}_{\mathsf{C}}(X, Y^E) \cong \underline{\mathsf{Hom}}_{\mathcal{E}}(E, \underline{\mathsf{Hom}}_{\mathsf{C}}(X, Y))$$

for each $X, Y \in \mathsf{C}$ and $E \in \mathcal{E}$.

**7.10.** Any symmetric monoidally closed category is enriched, tensored and cotensored over itself. Here we take the internal hom to define the tensor and cotensor.

**7.11.** Any locally small category $\mathsf{C}$ with small coproducts is tensored over sets, any category with small products is tensored over sets. In this case we have for $E \in \mathcal{S}\mathrm{et}$ and $X \in \mathsf{C}$:

$$E \cdot X := \coprod_{E} X, \qquad X^E := \prod_{E} X$$

**7.12.** An **monoidal model category** is an monoidal category $\mathsf{C}$ with a model structure such that

**monoidal model category**

- The category $\mathsf{C}$ is complete and cocomplete
- the monoidal structure is compatible with the model structure [Lur09, A.3.1.1].
- the monoidal structure is closed and symmetric.

**7.13.** Both $\mathcal{T}\mathrm{op}_q$ and $\mathrm{s}\mathcal{S}\mathrm{et}_q$ are monoidal model categories with the cartesian closed structure. Moreover the homotopical equivalence of 6.2 given by geometric realization $|-|$ and $\mathsf{Sing}$ preserves the monoidal structure. The right adjoint $\mathsf{Sing}$ preserves products automatically, for geometric realization this is a non trivial theorem (see [HM18, 2.6]).

**7.14.** Consider an $\mathcal{E}$-category $\mathsf{C}$ where $\mathcal{E}$ is an monoidal model category with localization map $[-] : \mathcal{E} \to \mathsf{Ho}(\mathcal{E})$, then the categorical structure is compatible with the model structure and so we can define it's homotopy category $h\mathsf{C}$ where

$$\mathsf{Hom}_{h\mathsf{C}}(A, B) := [\underline{\mathsf{Hom}}(A, B)]$$

**7.15.** The category of $\mathcal{E}$-categories $\mathcal{C}\mathrm{at}_{\mathcal{E}}$ where $\mathcal{E}$ is an monoidal model category inherits an homotopical structure in the following way. A functor $F : \mathsf{C} \to \mathsf{D}$ is an weak equivalence if $F(f) : \underline{\mathsf{Hom}}_{\mathsf{C}}(A, B) \to \underline{\mathsf{Hom}}_{\mathsf{D}}(F(A), F(B))$ is a weak equivalences for each $f$, $A$ and $B$. We will speak of the category of homotopically $\mathcal{E}$-categories when we mean the category of $\mathcal{E}$-categories with this homotopical structure.

## 7.2 Homotopy coherent diagrams

Suppose we have an ordinary category $\mathsf{A}$ and a homotopical category $\mathsf{C}$. Then an diagram of shape $\mathsf{A}$ in $\mathsf{C}$ is a functor $X : \mathsf{A} \to \mathsf{C}$. These functors assemble into a category of functors $\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{C})$. In homotopical spirit we should be free to replace objects $X(i) \in \mathsf{C}$ in the diagram $X : \mathsf{A} \to \mathsf{C}$ with weakly equivalent objects $Y(i)$. Such a replacement defines a natural weak equivalence $f : X \Rightarrow Y$ in $\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{C})$. These 'pointwise' weak equivalences define the class of weak equivalences of the category $\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{C})$. This is the case because these are precisely the natural transformations that become natural isomorphisms after postcomposition with $\gamma : \mathsf{C} \to \mathsf{Ho}(\mathsf{C})$. By localization we can form $\mathsf{Ho}(\mathcal{F}\mathrm{un}(\mathsf{I}, \mathsf{C}))$ and attempt to study invariant diagrams of shape $\mathsf{A}$.

When $\mathsf{C}$ is an simplical category with its canonically induced homotopical structure there is a very pleasing description of the homotopy classes of $\mathsf{Ho}(\mathcal{F}\mathrm{un}(\mathsf{I}, \mathsf{C}))$. There is a simplicial category $\mathfrak{C}A$ with the property that each $F \in \mathsf{Ho}(\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{C}))$ is represented by an ordinary simplical functor $\mathfrak{C}A \to \mathsf{C}$. The maps $\mathcal{F}\mathrm{un}(\mathfrak{C}\mathsf{A}, \mathsf{C})$ are called the **homotopy coherent diagrams** of shape $\mathsf{A}$ in $\mathsf{C}$.

**homotopy coherent diagrams**

**7.16.** Recall that there is a adjunction between $\mathcal{C}\mathrm{at}$ and $\mathsf{rGraph}$, the category of reflective graphs. This adjunction defines a comonad $T = FU$ on $\mathcal{C}\mathrm{at}$ which first sends a category to its underlying reflective graph and then produces the free category on this graph. In general one obtains from a comonad $(T, \mu, \epsilon)$ on $\mathsf{C}$ its **comonadic resolution functor** $\mathsf{res}_T : \mathsf{C} \to \mathcal{F}\mathrm{un}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{C})$. An object $A$ is sent to the simplicial object $\mathsf{res}_T(A) : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$ such that $[n] \mapsto T^{n+1}(A)$, the face maps are then derived from the comultiplication maps and the

**comonadic resolution functor**

degeneracy maps are derived from the counit maps as displayed below. This simplicial object can in fact be augmented such that $[-1] \mapsto A$, as also displayed.

$$\begin{tikzcd}
A & T(A) \arrow[l, "\epsilon"'] \arrow[r, "\mu"] & T^2(A) \arrow[l, shift right=2, "T\epsilon"'] \arrow[l, shift left=2, "\epsilon T"] \arrow[r, shift left=2, "\epsilon T"] \arrow[r, shift right=2, "T\epsilon"'] & T^3(A) \arrow[l, shift right=4, "T^2\epsilon"'] \arrow[l, "T\epsilon T"] \arrow[l, shift left=4, "\epsilon T^2"] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \cdots \arrow[l, shift right=6] \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[l, shift left=6]
\end{tikzcd}$$

**7.17.** When the comonad arises out of a free-forgetful adjunction $F \vdash U$ the simplicial object $\mathsf{res}_{FU}(A)$ is called the **free resolution** of $A$. For the adjunction between $\mathcal{C}\mathrm{at}$ and rGraph the category of reflective graphs write $\mathfrak{C} := \mathsf{res}_T : \mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{C}\mathrm{at}$ for resulting resolution functor.

**free resolution**

For a category A the free resolution $\mathfrak{C}\mathsf{A} : \mathbf{\Delta} \to \mathcal{C}\mathrm{at}$ is a cosimplicial object in $\mathcal{C}\mathrm{at}$. In fact, since for every $n$ we have $\mathcal{O}\mathrm{bj}(\mathfrak{C}\mathsf{A_n}) = \mathcal{O}\mathrm{bj}(\mathsf{I})$, $\mathfrak{C}\mathsf{A}$ is also a simplicial category in the sense that it is a category enriched in simplicial sets. This means that $\mathfrak{C}\mathsf{A}$ is presented by a set of objects and for every $n$ there is a category of $n$-arrows $\mathfrak{C}\mathsf{A}_n = (FU)^{n+1}A$ which consists of strings of composable arrows enclosed in sets of parenthesis with depth $n$. The counit maps then remove parenthesis by evaluation while comultiplication adds parenthesis by enclosing composed arrows.

**7.18.** Consider arrows $f, g, gf$ in the category A. Then in $\mathfrak{C}\mathsf{A}_0$ we find arrows such as $(f)$, $(g)$, $(gf)$, but also $(g) \circ (f)$. Clearly there should be some relation between $(g) \circ (f)$ and $(gf)$ if the unaugmented $\mathfrak{C}\mathsf{A}$ is to faithfully encode A. The relation between $(g)(f)$ and $(gf)$ is witnessed by $((g)(f)) \in \mathfrak{C}\mathsf{A}_2$ which, as the reader can verify, has exactly $(g)(f)$ and $(gf)$ as faces. One should think of the higher arrows in $\mathfrak{C}\mathsf{A}$ as homotopies relating the lower arrows.

The category A can also be considered as a constant simplicial category in the sense that $\mathsf{Hom}(A, B)_n = \mathsf{Hom}(A, B)$ and the face/degeneracy maps are all identities. This corresponds to a situation where the $n$-simplices of $\mathsf{Hom}(A, B)$ are just the degenerated ordinary arrows. The category A and $\mathfrak{C}\mathsf{A}$ are weakly equivalent objects in $\mathrm{s}\mathcal{C}\mathrm{at}$, the category of simplicial categories. This homotopical category inherits its structure from simplicial sets by saying that a functor $F : \mathsf{A} \to \mathsf{B}$ is a weak equivalence in $\mathrm{s}\mathcal{C}\mathrm{at}$ if it induces a weak equivalence for the hom-spaces. This is the case if the augmented simplicial object displayed above admits extra degeneracy maps $s_{-1} : A \to T(A)$, which are sections to the top face maps displayed above by [Mey84, Ch. 6].

**7.19.** The free resolution of a category $\mathfrak{C}\mathsf{A}$ provides a 'thickened up' version of A. It is in fact a cofibrant replacement for A in the model structure on the Bergner model structure of 7.31. In practice this means that composition has become 'homotopical' in the sense that a pair of composable morphisms $f, g \in \mathsf{A}$ which had to satisfy strict composition in A are now represented by $(f), (g), (gf) \in \mathfrak{C}\mathsf{A}_1$. The strict composition is then replaced by a the 2-arrow $((f)(g))$ with faces $(f)(g)$ and $(fg)$ which witnesses the composition. This makes it such that we can be more flexible when construction a simplicial functor $\mathfrak{C}\mathsf{A} \to \mathsf{C}$. No longer is there strict composition for the morphisms arising from A, instead requiring composition to be witnessed by a simplex (i.e. homotopy) in C.

**7.20.** Suppose we have an homotopical category C, then it is in general not the case that $\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{Ho}(\mathsf{C}))$ is equivalent to $\mathsf{Ho}(\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{C}))$ for the natural weak equivalence homotopy structure, defined above.

**7.21.** To give an example showing that these categories are not equivalent we exhibit an functor $F : \mathsf{C} \to \mathsf{Ho}(\mathcal{T}\mathrm{op})$ which can not be lifted to an homotopical functor into $\mathcal{T}\mathrm{op}$. For this the reader needs to know that there is a map $p : S^3 \to S^2$ called the Hopf fibration which is a fibre bundle with fibers $S^1$. This means that if we pick a basepoint in $S^2$ then the fiber over in $S^3$ is $S^1$ as displayed in the pullback below.

$$\begin{tikzcd}
S^1 \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S^3 \arrow[d, "p"] \\
* \arrow[r, "*"] & S^2
\end{tikzcd}$$

Expanding the diagram with an automorphism $n : S^1 \to S^1$ of degree $n$ we obtain a diagram which commutes in $h\mathcal{T}\mathrm{op}$ displayed below on the right. Indeed $in \sim i$ because all circles in $S^3$ are nullhomotopic and the rest of the diagram was already commutative in $\mathcal{T}\mathrm{op}$.

$$\begin{tikzcd}
S^1 \arrow[r, "i"] \arrow[d, "n"'] \arrow[dr, "*"] & S^3 \arrow[d, "p"] \\
S^1 \arrow[r, "*"] \arrow[ur, "i"'] & S^2
\end{tikzcd}
\qquad
\begin{tikzcd}
S^1 \arrow[dr, "n"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right] & & \\
& S^1 \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S^3 \arrow[d, "p"] \\
& * \arrow[r, "*"] & S^2
\end{tikzcd}$$

Now suppose that the diagram on the left is a homotopy coherent diagram; then we have a homotopy $\alpha : in \sim i$ such that $p\alpha : pin \sim pi$ is 2-homotopic to the constant homotopy. Since $p$ is a fiber bundle we can lift this homotopy to obtain a homotopy $\beta : in \sim i$ such that $p\beta$ is the constant homotopy at the basepoint. The homotopy $\beta$ shows that the map $n$ is obtained from the universal property of the homotopy pullback, but then $n \sim \mathsf{id}$ which only happens for $n = 1$.

**7.22.** In the category of categories $\mathcal{C}\mathrm{at}$ natural transformations between functors is represented using the directed interval $[1]$. A natural transformation between $F, G : \mathsf{C} \to \mathsf{D}$ is represented by a functor $\mathsf{C} \times [1] \to \mathsf{D}$. This suggests a natural extension to homotopy coherent functors.

**7.23.** An **homotopy coherent natural transformation** between homotopy coherent diagrams $F, G : \mathsf{A} \to \mathsf{C}$ (where $\mathsf{C}$ is a simplicial category) is given by an homotopy coherent diagram $\Delta[1] \times \mathsf{A} \to \mathsf{C}$ such that the following diagram commutes

**homotopy coherent natural transformation**

$$\begin{tikzcd}
\mathfrak{C}A \arrow[d, "\mathfrak{C}i_0"'] \arrow[dr, "F"] & \\
\mathfrak{C}\Delta[1] \times A \arrow[r] & \mathsf{C} \\
\mathfrak{C}A \arrow[u, "\mathfrak{C}i_1"] \arrow[ur, "G"'] &
\end{tikzcd}$$

**7.24.** The collection of ordinary functors $\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{C})$ assembles into a category since natural transformation can be uniquely composed. Two natural transformations $\alpha, \beta : [1] \times \mathsf{A} \to \mathsf{C}$ yield an unique $[2] \times \mathsf{A} \to \mathsf{C}$; this correspond by the cartesian closed structure to a functor $[2] \to \mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{C})$ or a triangle in $\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{C})$; two faces correspond to $\alpha$ and $\beta$ and the third face is the composite. Since this extension is unique it yields a composition rule and hence turns $\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{C})$ into a category.

Unfortunately there is no unique composite in the case for homotopy coherent functors. This is essentially because composition of homotopies is not unique. Instead we have to take all possible paths $[n]$ into account to reveal the structure of homotopy coherent natural transformations.

**7.25.** For any category $\mathsf{A}$ and simplicial category $\mathsf{C}$ let $\mathsf{Coh}(\mathsf{A}, \mathsf{C})$ denote the simplicial set of homotopy coherent diagrams of shape $A \times [n]$ for $n \geq 0$. This means that

- The 0-simplices $\mathsf{Coh}(\mathsf{A}, \mathsf{C})_0$ are homotopy coherent diagrams of shape $\mathsf{A}$ in $\mathsf{C}$.
- The 1-simplices $\mathsf{Coh}(\mathsf{A}, \mathsf{C})_1$ are homotopy coherent natural transformations.

**7.26.** Suppose $\mathsf{C}$ is a Kan-complex enriched category. In this case $\mathsf{Coh}(\mathsf{A}, \mathsf{C})$ has the right lifting property against inner horns (see 6.12). In other words, for all $0 < i < n$ the following lifts exist

$$\begin{tikzcd}
\Lambda^i[n] \arrow[r] \arrow[d, hook] & \mathsf{Coh}(\mathsf{A}, \mathsf{C}) \\
\Delta[n] \arrow[ur, dashed] &
\end{tikzcd}$$

see [Rie18].

## 7.3 ∞-categories

Higher categories generalize the notion of an ordinary category in that we add higher morphisms between morphisms, which are then themselves subject to be connected with higher morphisms and so on. Why would one want to do something like this? It turns out that one often naturally encounters categories for which the hom sets are themselves categories in their own right. This means that parallel morphisms in our category are objects in the hom category and can potentially be connected by a morphism.

The paradigmatic example is the category of categories $\mathcal{C}\mathrm{at}$ itself: the hom sets are the functor categories $\mathcal{F}\mathrm{un}(\mathsf{A}, \mathsf{B})$. We can try to axiomatize the properties of $\mathcal{C}\mathrm{at}$ to obtain the notion of a strict 2-category.

**7.27.** An **strict 2-category** $\mathsf{C}$ is a category enriched in $\mathcal{C}\mathrm{at}$.

**strict 2-category**

The collection of all strict 2-categories assemble into a category and we can then define a strict 3-category to be a category enriched in strict 2-categories. In this fashion we can define strict $n$-categories for all $n \geq 1$.

A problem the definition of a strict 2-category is that laws involving 1-morphisms are too strict. Indeed, if parallel 1-morphisms $f, g : A \to B$ are objects of a category $\mathsf{Hom}(A, B)$ then equality between such morphisms should be isomorphism in the category $\mathsf{Hom}(A, B)$.

The proper definition of a $n$-category is called a weak $n$-category, for $n = 2$ it is also called an **bicategory** as defined by Benabou [Bén67]. It can be thought of as an explicit description of categories weakly enriched in $\mathcal{C}\mathrm{at}$, i.e. we use the standard homotopical structure on $\mathcal{C}\mathrm{at}$. The full axiomatization of a bicategory is already involved and this only gets worse when ascending the ladder of (weak) $n$-categories for higher $n$.

In the limiting case we speak of an $\omega$-category. This is an category with $n$-morphisms for all $n \geq 0$ such that the structure at level $n$ is coherent only up to equivalence at level $(n+1)$ all the way up. This is stated imprecisely, and it turns out that making this precise requires a lot of work.

Luck has it that there is already one $\omega$-category that everyone is familiar with. To the space $X$ we can assign an $\omega$-category $\pi X$ in the following way. The objects of $\pi X$ are the points of $X$. Then 1-morphisms between points $x, y$ are given by paths $[0, 1] \to X$ with endpoints $x$ and $y$. The 2-morphisms are given by homotopies between such paths, 3-morphisms by homotopies between homotopies and so on.

This $\omega$-category is not strict: there is no way to pick a composition of paths that is strictly associative. Indeed if we have composable paths $f, g$ represented by morphisms $f, g : I \to X$ we could define the composite path $gf : x \to z$ to be given by

$$fg : t \mapsto \begin{cases} f(2t) & \text{if } t < \frac{1}{2} \\ g(2t-1) & \text{if } t \geq \frac{1}{2} \end{cases}$$

But this composition is not associate, the composites $h(gf)$ and $(hg)f$ behave much differently.

The $\infty$-categories $\pi X$ for topological spaces $X$ are actually instances of **$\infty$-groupoids**, these are $\infty$-categories such that all $n$-morphisms are invertible. Without properly defining yet what an $\infty$-category is we already have a class of objects modeling a special case. The idea of the next chapter is to take the $\infty$-groupoids modeled by spaces as a starting point for defining proper $(\infty, 1)$-categories.

**$\infty$-groupoids**

If we only care about the $\infty$-groupoid presented by a space it makes sense to consider $\mathcal{T}\mathrm{op}_q$ with the Quillen model structure. Indeed two categories are weakly equivalent when they present the same $\infty$-groupoid. This is because the $\infty$-groupoid encodes all the higher homotopy groups at every basepoint of $X$.

A priori it is not clear whether every $\infty$-groupoid is modeled by a space, the assertion that this is the case is Grothendiecks **homotopy hypothesis**. More formally it asserts that $\infty$-groupoids are equivalent to topological spaces up to weak homotopy equivalence. For us this hypothesis will be a definition.

**homotopy hypothesis**

An $\infty$-groupoid should be an $\omega$-category such that all $n$-morphisms for $n > 0$ are invertible. It is useful to classify higher categories by the level at which all higher arrows are invertible. Let an $(\infty, m)$-category be an $\omega$-category such that all $n$-morphisms for $n > m$ are invertible. Since $\omega$-categories are not properly defined this definition should be used more as an informal guide.

Our goal this section will be to introduce some models of $(\infty, 1)$-categories which we will just call $\infty$-categories. Following the ideas presented above, i.e. that

- higher categories can be obtained by enriching categories in lower categories;
- and spaces up to weak equivalence $\mathcal{T}\mathrm{op}_q$ serves as a model for $\infty$-groupoids;

lead to a natural suggestion for such an model.

**7.28.** An model of $\infty$-categories is given by the homotopical category of topologically enriched categories. This are the $\mathcal{T}\mathrm{op}$-enriched categories which we will call **topological categories** equipped with the homotopical structure where a functor $F : \mathsf{C} \to \mathsf{D}$ between topological categories is an **equivalence** if it is an weak equivalence on all the hom sets, as defined in 7.15. The topological categories with this homotopical structure will be written as $\mathrm{t}\mathcal{C}\mathrm{at}_b$.

**topological categories**

**7.29.** The above definition is perhaps the most intuitive definition of an $\infty$-category. However it is rather difficult to work with in practice. This means we will look for different models of $\infty-$categories.

**7.30.** The adjunction between $\mathcal{T}\mathrm{op}$ and $\mathrm{s}\mathcal{S}\mathrm{et}$ descends to $\mathcal{T}\mathrm{op}$-categories and $\mathrm{s}\mathcal{S}\mathrm{et}$-categories because $\mathsf{Sing}$ and $N$ both preserve finite limits (i.e. the monoidal structure in question). Indeed if we have an $\mathcal{T}\mathrm{op}$-category we obtain an $\mathrm{s}\mathcal{S}\mathrm{et}$-category by applying $\mathsf{Sing}$ to all the hom spaces and vice versa.

**7.31.** Another model of $\infty$-categories is given by the homotopical category of simplicialy enriched categories i.e. $\mathrm{s}\mathcal{S}\mathrm{et}$-enriched categories, we will call such categories **simplicial categories**. This category is also given the homotopical structure derived from $\mathrm{s}\mathcal{S}\mathrm{et}_q$ in a similar way to $\mathrm{t}\mathcal{C}\mathrm{at}_b$ above. This homotopical structure is called the Bergner structure and we will write it as $\mathrm{s}\mathcal{C}\mathrm{at}_b$. Bergner extended this model structure to an full model structure in [Ber04].

**simplicial categories**

**7.32.** The homotopical equivalence between $\mathrm{s}\mathcal{S}\mathrm{et}_q$ and $\mathcal{T}\mathrm{op}_q$ from 6.2

$$\begin{tikzcd}
\mathrm{s}\mathcal{S}\mathrm{et}_q \arrow[r, bend left, "|-|_{\mathcal{T}\mathrm{op}}"] \arrow[r, phantom, "\perp"] & \mathcal{T}\mathrm{op}_q \arrow[l, bend left, "\mathsf{Sing}"]
\end{tikzcd}$$

induces an adjoint equivalence between simplicial categories and topological categories given by applying $\mathsf{Sing}/|-|_{\mathcal{T}\mathrm{op}}$ to every hom object.

**7.33.** The above adjoint homotopical equivalence between topological categories and simplicial categories ensures that we can replace a topological category $\mathsf{C}$ with an equivalent $\mathsf{C}'$ enriched in CW-complexes by setting $\mathsf{Hom}_{\mathsf{C}'}(X, Y) = |\mathsf{Sing}\,\mathsf{Hom}_{\mathsf{C}}(X, Y)|$. Analogously, we take every simplicial category to be enriched in Kan complexes.

**7.34.** Our last model of $\infty$-categories will be quasi-categories. These are simplicial sets having the (weak) lifting property against the inner horns from 6.12. This means that a simplicial set $\mathsf{C}$ is an quasi-category if for all $n$ and $0 < i < n$ and any map $f$ there is a dashed lift

$$\begin{tikzcd}
\Lambda^i[n] \arrow[r, "f"] \arrow[d, hook] & \mathsf{C} \\
\Delta[n] \arrow[ur, dashed] &
\end{tikzcd}$$

**7.35.** The adjunction between categories and simplicial sets of 6.8 shows that categories can be faithfully encoded as certain simplicial sets. We saw that the essential image of the nerve $N : \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}$ can be characterized as those simplicial sets enjoying a unique right lifting property against the inner horns from 6.12.

**7.36.** The idea is to upgrade the nerve-realization adjunction between $\mathcal{C}\mathrm{at}$ and $s\mathcal{S}\mathrm{et}$ to be defined on $s\mathcal{C}\mathrm{at}$. We apply extension by colimits to the functor $\Delta \to s\mathcal{C}\mathrm{at}$ given by the map $[n] \mapsto \mathfrak{C}([n])$ where $\mathfrak{C}$ is the free resolution of the category $[n]$. We will call this extension $\mathfrak{C} : s\mathcal{S}\mathrm{et} \to s\mathcal{C}\mathrm{at}$. This is potentially ambiguous but we will show that the difference is inconsequential in 7.37.

$$\begin{tikzcd}
s\mathcal{S}\mathrm{et} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & s\mathcal{C}\mathrm{at} \arrow[l, bend left, "N"]
\end{tikzcd}$$

The right adjoint is the **homotopy coherent nerve** and sends an simplicial category $\mathsf{C}$ to the simplicial set $N(\mathsf{C}) : [n] \mapsto \mathsf{Hom}(\mathfrak{C}[n], \mathsf{C})$.

**homotopy coherent nerve**

**7.37.** *Our previous definition* $\mathfrak{C} : \mathcal{C}\mathrm{at} \to s\mathcal{C}\mathrm{at}$ *agrees with* $\mathfrak{C} : s\mathcal{S}\mathrm{et} \to s\mathcal{C}\mathrm{at}$ *in the sense that the diagram below commutes*

$$\begin{tikzcd}
\mathcal{C}\mathrm{at} \arrow[r, "\mathfrak{C}"] \arrow[d, "N"] & s\mathcal{C}\mathrm{at} \\
s\mathcal{S}\mathrm{et} \arrow[ur, bend right, "\mathfrak{C}"'] &
\end{tikzcd}$$

*Proof.* For more detail see [Rie18]. □

**7.38.** The map $\mathfrak{C}$ allows us to transport the homotopical structure $s\mathcal{C}\mathrm{at}_b$ to $s\mathcal{S}\mathrm{et}$. The resulting homotopical structure is the Joyal homotopical structure and is written as $s\mathcal{S}\mathrm{et}_j$. Andre Joyal extended this homotopical structure to a model structure (see [Lur09, 2.2.5]). An weak equivalence in the Joyal homotopical structure on $s\mathcal{S}\mathrm{et}$ is called an **categorical equivalence**, i.e. a map $f : A \to B$ of simplicial such that the induced map $\mathfrak{C}f : \mathfrak{C}A \to \mathfrak{C}B$ is an equivalence in $s\mathcal{C}\mathrm{at}_b$.

**categorical equivalence**

**7.39.** The idea is now to show that $s\mathcal{S}\mathrm{et}_j$ and $t\mathcal{C}\mathrm{at}_b$ are homotopically equivalent. For this we use $s\mathcal{C}\mathrm{at}_b$ as an intermediate to obtain the composite adjunction as follows

$$\begin{tikzcd} \mathrm{s}\mathcal{S}\mathrm{et} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathrm{s}\mathcal{C}\mathrm{at} \arrow[l, bend left, "N"] \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathrm{t}\mathcal{C}\mathrm{at} \arrow[l, bend left, "\mathsf{Sing}"] \end{tikzcd}$$

We will call the right adjoint the **topological nerve** and write it as $N$. We get a composed adjunction $|\mathfrak{C}(-)] \dashv N$. A map $f : S \to T$ in $\mathrm{s}\mathcal{S}\mathrm{et}$ is a categorical equivalence if and only if $\mathfrak{C}F : \mathfrak{C}S \to \mathfrak{C}T$ is an equivalence of simplicial categories if and only if $|\mathfrak{C}F| : |\mathfrak{C}S| \to |\mathfrak{C}T|$ is an equivalence of topological categories. Then the unit of the adjunction is an equivalence iff $|\mathfrak{C}(\eta)|$ is an weak equivalence. By the triangle law the following commutes

**topological nerve**

$$\begin{tikzcd} |\mathfrak{C}(S)| \arrow[r, "|\mathfrak{C}(\eta_S)|"] \arrow[dr, "\mathsf{id}"'] & |\mathfrak{C}(N(|\mathfrak{C}(S)|))| \arrow[d, "\epsilon_{|\mathfrak{C}(S)|}"] \\ & |\mathfrak{C}(S)| \end{tikzcd}$$

in which id is a weak equivalence. By two-out-of-three it is enough to show that the counit maps are weak equivalences. For this we need the following result:

**7.40.** *If* $\mathsf{C}$ *is an Kan enriched category, i.e. an simplicial category such that the mapping spaces are all Kan complexes then*

*(i) the counit map* $\mathit{Map}_{\mathfrak{C}[N\mathsf{C}]}(X, Y) \to \mathit{Map}_{\mathsf{C}}(X, Y)$ *is a weak equivalence.*

*(ii) the nerve* $N\mathsf{C}$ *is an* $\infty$*-category.*

*Proof.* For (i) see [Lur09, 2.2.0.1] and for (ii) see [Lur09, 1.1.5.10] □

**7.41.** It is enough to see that for a topological category $\mathsf{C}$ the associated simplicial category obtained by applying $\mathsf{Sing}$ to the hom spaces is a Kan enriched category. We therefore have an equivalence between the two models of $\infty$-categories $\mathrm{t}\mathcal{C}\mathrm{at}_b$ and $\mathrm{s}\mathcal{S}\mathrm{et}_j$. After we also take the equivalence between $\mathrm{t}\mathcal{C}\mathrm{at}_b$ and $\mathrm{s}\mathcal{C}\mathrm{at}_b$ into account we obtain the following result.

(i) every topological category is equivalent to an CW-enriched category.

(ii) every simplicial set is categorically equivalent to an quasi-category.

(iii) every simplical category is equivalent to a Kan enriched category.

There are many different models of $\infty$-categories, luckily the above shows that they are all equivalent. Nonetheless it is good to keep in mind different perspectives.

# Index

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